Residual Model for Future Prices

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Abstract

This paper presents a new factor model for the term structure of futures prices of commodities. This model fills a gap in the literature by providing not only flexibility on the deterministic drivers of the term structure's (TS) curve but also a clear meaning of the stochastic factors implied by the model. These benefits allow the user of the model to predict, and to protect himself against, changes in the slope and concavity of the TS curve. In particular, these new factors are identified as the spot price, the slope and the concavity of the curve, and they directly tackle the phenomena defined as contango and backwardation movements. It is shown that the model provides a good fit for the term structure's curve under the historical measure. Conditions on the processes of the factors are obtained under an equivalent measure which ensures an absence of arbitrage. Explicit expressions for the term structure of volatilities and correlations are also provided. The paper concludes with a set of derivatives inspired by the new factors which could be used to protect an investor against contango and backwardation movements.

Keywords: Future prices, Term structure, Arbitrage measure, Contango, Backwardation

1. Introduction

The stochastic behaviour of spot and futures prices for commodities is of paramount importance to analysts interested in several important fields in finance, such as risk management and the pricing of contingent claims (derivatives). The modelling of commodity prices is one of the oldest and most challenging problems faced by practitioners and academics in the financial industry. Commodity prices present a variety of features which make them a very special case in the financial market. They show a very flexible term structure (TS) of futures prices $(F_{t,T})$, where $(F_{t,T})$ represents the price of the commodity at T as seen at t, where $t \le T$, which is far richer than the term structure of bond prices. Note that from an empirical viewpoint, the term structure of bond prices ($B(t,T)$: the value at t of 1 dollar paid at T) should always be decreasing in $T-t$ so, for example, changes in slope and curvature in time to maturity, $T-t$, are quite common in commodities while almost unobservable in bonds. This implies a higher level of complexity for risk management and derivative pricing purposes. The objective of this paper is to provide a flexible model for the term structure of commodities, based on a polynomial fitting of the TS curve. This fitting leads to clear new tradable factors which could potentially protect investors against movements in the TS curve. This model is inspired by Garbade (1996), who focused on bond prices.

The first step in the modelling process is an understanding of the practical features of the available data. The future price of a commodity, at time t, can be observed for fixed maturity times T_i (time to maturity $T_i - t$). For example, in the case of oil, T_i would correspond to the 20th of the month, with a $T_i - t$ of up to several years. The most liquid contracts are those with maturity of up to twelve months, but nowadays contracts for up to seven years have become more common. Consequently, the prices of futures contracts with maturities greater than 12 or 18 months are frequently only an extrapolation of the prices of shorter maturity contracts computed from actual trades.

The standard modelling of this vector of data $(F_{t,T_1},...,F_{t,T_n})$ is based on the assumption that F_{t,T_i} is a point of a curve on R^2 for each time t (with this curve being known as the term structure curve or futures prices curve). Then the objective is to model the whole surface $F_{t,T}$ as a function of several stochastic factors with respect to t as well as deterministic drivers with respect to T.

For every commodity, the exercise of modelling involves two possible stochastic behaviours associated with the term structures. One stochastic model is based on calibrating the surface $F_{t,T}$ using a history of observable 'curves' $F_{t_1,T},\ldots,F_{t_m,T}$ (here $T=(T_1,\ldots,T_n)$, and $t_1<\ldots<t_m<s$, where s is today). This stochastic model is called the historical model, and the probability measure underlying it, which is used to forecast $F_{t,T}$ (with $t>s$), is called the historical measure, or P-measure. This model is mostly used for Risk Management purposes, i.e. optimization and VaR (Value at Risk) calculations of portfolios on spot and futures prices.

The second main objective in finance is to use these prices (spot and futures) as the underlying assets for derivative pricing purposes (to obtain the present value of contingent claims on those underlying assets, i.e. European options on spot or futures prices). In such a case, the probability measure describing the future behaviour of these curves $F_{t,T}$ (with $t>s$) is different from the historical measure, as it has to account for arbitrage opportunities (see Bjork (2002)). This measure is called the Risk-Neutral Measure or Arbitrage-free measure (from now on, the Q-measure).

Early studies in modelling typically assumed that all uncertainty arises from the spot price of the commodity $F_{t,t}$. This is the model of stock price uncertainty underlying the famous Black and Scholes option pricing formula, and it leads to closed-form solutions for many derivatives prices and to simple VaR calculations for Risk Management (see for example, Schwartz (1982) and Brennan and Schwartz (1985)). Recognition of the importance of the variability of the spreads between spot and futures prices (term structure) led to the development of several multi-factor models of commodity prices. For example, Gibson and Schwartz (1990) introduced a two-factor model where the spot price of the commodity and the convenience yield, defined as the flow of services that accrues to the holder of the physical commodity, but not to the owner of a contract for future delivery, followed a joint stochastic process. Schwartz (1997) presented a three-factor model where the logarithm of the spot price, the convenience yield and interest rates followed mean reverting processes.

Based on the fact that many find the notion of convenience yield elusive, Schwartz and Smith (2000) proposed a different approach to commodity modelling. They considered a model with two factors, called short term and equilibrium term. Although this model is equivalent to the Gibson-Schwartz (1990) model, it leads to analytic results that are more transparent and allows a simplification of the analysis of long-term investment. Pilipovic (1998) has also developed bi-factor price models. On a different path, Vidal et al. (1999) added complexity to the model, by assuming stochastic volatility and correlation among factors, at the expense of even less clarity of meaning for the drivers and factors of the model. Moreover, the papers by Tolmasky and Hindanov (2002), Lautier (2005), and Jaimungal and Ng (2008) show recent attempts to apply Principal Component Analysis to the term structure of futures prices, while Gorton and is Rouwenhorst (2006) provide a detailed review of the behaviour of the curve.

One of the prominent features of most term structures for commodities (such as energy, agricultural products and metals) are the sudden changes in sign of their slope and convexity, which translate into interesting movements of the futures prices curve. This behaviour is generally known as backwardation (when decreasing) and contango (when increasing). The actual definition of contango is a market state where futures prices are above the expected spot prices and fall as maturity approaches, while backwardation is a market state where futures prices are below expected spot prices and rise as maturity approaches. Models in the literature do not explicitly (though they do this implicitly) address this problem, as they do not allow a flexible set of meaningful underlying factors/drivers to be used, i.e. none of the factors driving the curve relates to the notion of slope or concavity of the curve. The main reason for this lack of flexibility in relation to the factors is the arbitrage-free constraints impose by the Q-measure. Under the Q-measure, the drivers associated with the factors must satisfy a system of differential equations with few closed-form expressions. These constraints do not apply to the historical measure.

This is why the finding of a flexible set of factors, under both measures, capable of explaining important features (movements) of the term structure of futures prices is the main focus of this paper. In particular, this paper studies a polynomial fitting of the term structure under the historical and risk neutral measures for oil futures prices in the American market.

This paper is organized as follows. In section 2, a three-factor model is developed under the historical measure (P-measure); these new factors are studied and show simple stochastic behaviour and second-moment dependency structures. The estimation of the model parameters and factors, as well as the empirical findings, are shown in subsection 2.2. The risk neutral (Q-measure) processes for these underlying factors are found in section 3. Several examples of contingent claims (derivatives) based on these factors are provided in section 4, which discusses hedging against contango and backwardation. Section 5 concludes.

The developments in these sections attempt to provide a complete picture of the two leading problems of the commodity world, which are forecasting, and pricing of commodity variables.

2. Historical Measure Model

The development of risk management methodologies for term structures (e.g. futures prices and bond prices) relies on the assumption that some chosen underlying market factors follow standard diffusion processes (such as geometric Brownian motion or mean reverting processes). Finding suitable sets of risk factors and/or hand-made relationships between a curve and the factors would appear to be a strong way to capture and handle complex behaviour in the curve.

On the other hand, while most efforts have been devoted to the explanation of the futures prices term structure in a risk-neutral world (using the Q-measure), little has been done regarding the term structure behaviour under the P-measure. The modelling with this measure is vital for the effective risk management of financial portfolios. For example, if one were interested in computing the h-days-horizon Value at Risk of a portfolio made of futures prices derivatives, one would need to model the underlyings (the futures prices) using historical data on futures prices; otherwise one would not be aware of the real risk of the portfolio, but only of the artificial risk from the Q-measure. In this paper a multi-factor model for futures prices is studied explicitly under the historical measure.

For the next sections, let us assume that there is a filtered probability space, $(\Omega, F, (F_t)_{t\geq 0}, P)$. We also assume a finite time horizon T^* with $F = F_{T^*}$. All definitions and statements are understood to be valid only until this time horizon T^*. Let $E_t^P[\bullet]$ denote the conditional expectation under the P-measure which is conditional on the information at date t, F_t. All equations between stochastic variables are to be understood as almost surely equations under the given probability measure.

2.1 A Proposed Model

The model for $F_{t,T}$ should belong to a family capable of making a good fit to the observable curve (for any given time t). This property should hold regardless of the measure considered – the measure could be either the historical measure or the measure implied by the absence of arbitrage. In other words, a curve is fitted at every t by using a convenient function of $T-t$, for example a polynomial:

$$F_{t,T} = S_t + \xi_t \cdot (T-t) + \eta_t \cdot (T-t)^2 + \chi_{t,T}, \tag{1}$$

The model is specified by the coefficients (factors) $\chi_{t,T}, S_t, \xi_t, \eta_t$ (stochastic processes on t), which may be seen as the parameters of a linear regression ($\chi_{t,T}$ residual), or the Taylor expansion of futures prices in terms of $T-t$:

$$F_{t,T} = F_{t,t} + \tfrac{\partial F_{t,T}}{\partial T}|_{T=t} (T-t) + \tfrac{\partial^2 F_{t,T}}{\partial^2 T}|_{T=t} (T-t)^2 + \chi_{t,T} \quad .$$

The two most important reasons for this proposal are first the meaning of the underlying factors (spot price, slope and concavity/convexity of the curve (contango and backwardation movements)) and secondly, the fact that the model transforms the given term structure $F_{t,T}$ into a simpler one $\chi_{t,T}$.

The factors are assumed to follow general diffusion processes:

$$\begin{aligned} d\chi_{t,T} &= f_\chi(S_t, \xi_t, \eta_t, t, T)dt + \sigma_{0,t,T} dW_t \\ dS_t &= f_S(S_t, \xi_t, \eta_t, t)dt + \sigma_{1,t} dW_t \\ d\xi_t &= f_{S_t,\xi}(\xi_t, \eta_t, t)dt + \sigma_{2,t} dW_t \\ d\eta_t &= f_\eta(S_t, \xi_t, \eta_t, t)dt + \sigma_{3,t} dW_t \end{aligned} \tag{2}$$

Here W_t is a d-dimensional vector of independent Brownian motions. The correlations come via the specification of the diffusion volatilities. The integrands $(f_\chi, f_S, f_\eta, f_\xi)$, $(\sigma_{0,t,T}, \sigma_{i,t})$ are predictable processes that are regular enough to allow for:

- Differentiation under the integral sign.
- Interchange of the order of integration.
- Partial derivatives with respect to the T-variable.
- Bounded for almost all events, $w \in \Omega$.

In particular, constant volatilities and the following drifts are proposed:

$$\begin{aligned} f_S &= k_S(\theta_{S,t} - S) \\ f_\eta &= k_\eta(\theta_{\eta,t} - \eta) \\ f_\xi &= k_\xi(\theta_{\xi,t} - \xi) \\ f_\chi &= a_0 + a_1 \cdot (T-t) + a_2 \cdot (T-t)^2 \end{aligned} \tag{3}$$

There are two reason for selecting this model; one comes from the empirical fitting to observable futures prices curves (see section 2.2), and the second is that it allows one to compute the market price of risk and thus to determine these processes completely in the absence of arbitrage (see section 3).

Note that this framework deals with a multivariate normal distribution, at any time t, instead of a lognormal for the futures prices. Therefore maturity times should be kept within the observable range (up to 9 months, which is what is observable) in order to avoid negative futures prices. The strength of this framework lies in the superior fitting (smaller errors and better forecasting, see section 2.2) of the futures prices term structure compared to other frameworks, as well as the meaning of the factors.

2.2 Empirical Results

One of the difficulties in the empirical implementation of futures prices models is that the factors or state variables of these models are not directly observable. For some commodities, the spot price is hard to obtain, and the futures contract closest to maturity is used as proxy for the spot price. The problems of estimating the instantaneous convenience yield are even more complex; normally, futures prices with different maturities are used to compute it. The instantaneous interest rate is also not directly observable.

In this section, the estimation procedure for our model is addressed. In this case, as in most practical cases, the underlyings are not directly observable (a proxy can be obtained by using a Taylor expansion interpretation), so they would also have to be estimated. A robust method for estimating factors and parameters is a two-step procedure based first on a linear regression to estimate the factors and then on a maximum likelihood estimation of the factor's parameters (this is equivalent to Kalman-Filter techniques, see Schwartz and Smith (2000) for a description).

A set of twelve time series of oil futures prices from January 1990 until December 2005 from the American Future Market is used; each series corresponds to a specific maturity month e.g. there are futures prices for $T_i - t$, (i=January,..., December), the time to maturity for each time t. A fitting of the future curve by a three-factor model is performed in this section. This fitting allows us to study the negligibility of the residual factor.

Based on model (1), a regression of the futures price curves (up to one year, which are actually the observable ones) is performed on a quadratic polynomial. The dependent variable is the maturity time T:

$$F_{t,T} = S_t + \xi_t \cdot (T-t) + \eta_t \cdot (T-t)^2 + \varepsilon(t,T), \tag{4}$$

This analysis shows the insignificance of the residual factor $\chi_{t,T}$. Note that $\varepsilon(t,T)$ is a good proxy for the residual $\chi_{t,T}$ from settings 2 and 3. The R-square of a quadratic polynomial fitting is obtained from the regressions. The R-square is a widely accepted measure of the quality of a least-squared fit. The better the fit, the more insignificant the error term $\varepsilon(t,T)$.

The fitting was performed to 770 futures prices curves $(F_{t,T_1}, \ldots, F_{t,T_{12}})$, with dates ($t$) from 1991 to 2005. The following table shows several statistics of the R^2 (correlation coefficient) obtained from those fittings (QP-quadratic polynomial).

Insert Table 1

The average of the R^2 was 0.97, but more importantly the 95 per cent confidence interval $[0.88, 0.999]$ for the value of R^2 implies a good fitting. Thus, the residual is basically insignificant in the explanation of the futures prices. A high correlation was detected between the linear and the quadratic term (0.83).

3. Arbitrage-free Model

As previously mentioned, the stochastic processes for the factors depend on the objective: risk management was covered in the previous section, while pricing derivatives is the focus of this section.

Derivative pricing is always a case of determining the price today, t, of a given function of the evolution of several underlying price factors, based on no-arbitrage conditions. The main focus is on the particular case where a specific period for this evolution $[t,T]$ is given (a contingent claim with date of maturity T). The stochastic process for the underlying factors, from t to T, in the absence of arbitrage, may be different from the one obtained by any sound statistical tools using historical (previous to t) data. This is why the key objective in derivative pricing is finding the factor's stochastic processes in the absence of arbitrage between t and T.

3.1 Equivalent Measure

It has been already shown that, under the historical measure, the initial curve has a polynomial shape (low order) on the time to maturity. In this section, model (1) is assumed for the futures prices, and then the conditions on the underlying factors $(\chi_{t,T}, S_t, \xi_t, \eta_t)$ under the arbitrage-free measure are provided.

The specification of the drifts for the underlying processes in the absence of arbitrage follow from the standard arbitrage argument: the futures prices process must satisfy the following condition: $F_{s,T} = E_s^Q[F_{t,T}]$ under the Q -measure. In words, the futures prices are a martingale in the measure induced by bond prices as numeraire (and Bjork (2002)). The change of measure from P to Q, which makes futures prices driftless, is induced by Girsanov's theorem. This would imply a drift change on the processes involved for each of the factors in the decomposition of the futures prices (1), while the volatility remains constant under a change of measure.

The following theorem provides the processes for the factors in the absence of arbitrage.

Theorem 1: *Assume the following model for the futures prices (model 1):*

$$F_{t,T} = S_t + \xi_t \cdot (T-t) + \eta_t \cdot (T-t)^2 + \chi_{t,T}, \tag{5}$$

where the factors follow diffusion under the historical measure:

$$\begin{aligned} d\chi_{t,T} &= f_\chi(\xi_t, \eta_t, t, T)dt + \sigma_{0,t,T}dW_t, \quad dS_t = f_S(\xi_t, \eta_t, t)dt + \sigma_{1,t}dW_t \\ d\xi_t &= f_\xi(\xi_t, \eta_t, t)dt + \sigma_{2,t}dW_t, \quad d\eta_t = f_\eta(\xi_t, \eta_t, t)dt + \sigma_{3,t}dW_t, \end{aligned} \tag{6}$$

Here W_t *is a d-dimensional (* $d > 2$ *) vector of independent Brownian motions.*

$$\begin{aligned} f_S &= k_S(\theta_{S,t} - S), \\ f_\xi &= k_\xi(\theta_{\xi,t} - \xi), \\ f_\eta &= k_\eta(\theta_{\eta,t} - \eta), \\ f_\chi &= \textstyle\sum_{i=0}^{d-1} a_i \cdot (T-t)^i, \\ \sigma_{0,t,T} &= \textstyle\sum_{i=0}^{d-1} b_i \cdot (T-t)^i, \end{aligned} \tag{7}$$

The k_i *are constants,* θ_i *are time dependent functions and* $a_i, b_i, \sigma_{0,t,T}, \sigma_{i,t}$ *,* $i = 1,2,3$ *) are* $1 \times d$ *time dependent vectors. The volatilities (* $\sigma_{i,t}$, $i = 0,\ldots,m$ *) are measurable and fulfil Lipschitz and growth conditions (such that a unique solution of the stochastic differential equations exists).*

Let us assume that the $d \times d$ *matrix* $\sigma(t)$ *is invertible for all* t*, where* $\sigma(t)$ *is defined by the following equation:*

$$\sum_{i=0}^{d-1} b_i \cdot (T-t)^i - \sigma_{1,t} - \sigma_{2,t} \cdot (T-t) - \sigma_{3,t} \cdot (T-t)^2 = \sigma(t) \cdot (1, T, ?, T^{d-1})' \tag{8}$$

Then there exists a unique market price of risk vector λ*, and, thus, a unique martingale measure* Q*, such that the futures prices are arbitrage free. Moreover, the factors' stochastic processes in the absence of arbitrage (under the* Q *-measure) are the following:*

$$\begin{aligned} dS_t &= g_{1,t}dt + \sigma_{1,t}dW_t^Q \\ d\xi_t &= g_{2,t}dt + \sigma_{2,t}dW_t^Q \\ d\eta_t &= g_{3,t}dt + \sigma_{3,t}dW_t^Q \\ d\chi_{t,T} &= (\xi_t + 2\eta_t(T-t) - g_{1,t} - g_{2,t}(T-t) - g_{3,t}(T-t)^2)dt + \sigma_{0,t,T}dW_t^Q, \end{aligned} \tag{9}$$

where $g_{1,t} = f_S - \lambda \cdot \sigma_{1,t}$, $g_{2,t} = f_\xi - \lambda \cdot \sigma_{2,t}$ *and* $g_{3,t} = f_\eta - \lambda \cdot \sigma_{3,t}$ *denote the difference between the drift under the* P *-measure and the market price of risk, the d-dimensional vector* λ *. Moreover* g_i *are linear functions of the factors.*

Proof: Let us denote the process under the P -measure for the futures prices as:

$$dF_{t,T} = \mu_F(t,T)dt + \sigma_F(t,T)dW_t, \tag{10}$$

where W_t is a d-dimensional vector of independent Brownian motions. As shown by Harrison and Pliska (1981) and Duffie (1996), the absence of arbitrage in the futures contract market is equivalent to the existence of a d

-dimensional column vector process $\left(\lambda(t),...,\lambda_d(t)\right)'$ (market price of risk), such that $\forall t$, $0<t<T$,

$$\mu_F(t,T) = -\sigma_F(t,T)\cdot\lambda(t) \tag{11}$$

This lambda vector comes from Girsanov's theorem, so the change of measure would be:

$$dW_t = dW_t^Q + \lambda(t)dt \tag{12}$$

Note that in terms of the Radon-Nikodym derivative, $\frac{dQ}{dP} = \exp \lambda_t \cdot W_t^Q - 1/2\lambda_t^2 \cdot t$, which is a more common way to present the change of measure. For convenience denote $X_{t,T} = (\chi_{t,T}, S_t, \xi_t, \eta_t)$.

Applying Ito's lemmait follows that:

$$\begin{aligned}\mu_F(t,T) &= -\xi_t - 2\eta_t(T-t) + f_S + (T-t)f_\xi + (T-t)^2 f_\eta + f_\chi \\ \sigma_F(t,T) &= \sigma_0 - \sigma_1 - \sigma_2(T-t) - \sigma_3(T-t)^2,\end{aligned} \tag{13}$$

Let us express the drift and volatility of the futures prices as follows:

$$\begin{aligned}\mu_F(t,T) &= \mu(t,X)\cdot\mathbf{T} \\ \sigma_F(t,T) &= \mathbf{T}\cdot\sigma(t),\end{aligned} \tag{14}$$

where $\mathbf{T}$ is a $1\times d$ vector of powers of T, $\sigma(t)$ is a $d\times d$ matrix and $\mu(t,X)$ is a $1\times d$ vector. The no-arbitrage condition (11) and the previous expressions for the drift and volatility of the futures prices and the residual (13 and 214 provide a system of equations for λ where the coefficients of the polynomial in T, ($1, T, T^2, ..., T^{d-1}$) have to be zero. Therefore equation (17) becomes:

$$(\lambda\cdot\sigma(t) - \mu(t,X))\cdot\mathbf{T} = 0 \tag{15}$$

which implies:

$$\begin{aligned}\lambda\cdot\sigma(t) &= \mu(t,X) \\ \lambda &= \mu(t,X)\cdot\sigma^{-1}(t)\end{aligned} \tag{16}$$

Note that λ would be a linear function of the factors (because $\mu(t,X)$ is linear).

The condition of no-arbitrage (11 (zero drift under the equivalent measure for the futures prices), together with equation (19), imply the following relationship between the drift of the residual and the other factors under Q:

$$g_R = \xi_t + 2\eta_t\cdot(T-t) - g_1 - (T-t)\cdot g_2 - (T-t)^2\cdot g_3 \tag{17}$$

where the drift of the stochastic processes for the factors are $g_i = f_i - \lambda * \sigma_i$, and the drift for the residual is g_χ.

Remark *Note that the previous result applies as long as ($\theta_i, k_i, a_i, b_i, f_\chi, \sigma_{0,t,T}$, $\sigma_{i,t}$, $i=1,2,3$) are measurable and fulfil Lipschitz and growth conditions, thus they are allowed to be functions of t, S, ξ, η. In this case, the factors' drifts $g_i, i=1,2,3$ under Q might not be linear on the factors. Nevertheless σ_t, which will now depend on the factors as well as time, is required to be invertible. Secondly, having calibrated the processes under P, the processes under Q are easily obtained by simply computing λ from the system of equations (16) and then g_1, g_2 and g_3 follow.*

This result can be extended by considering any kind of drivers $A_i(T-t)$:

$$F_{t,T} = \sum_{i=0}^{m} A_i(T-t)\cdot\chi_{i,t} + \chi_{t,T} \tag{18}$$

The previous theorem remains intact, with the exception that the stochastic processes for factors and residual under the Q -measure would be as follows:

$$\begin{aligned}d\chi_{i,t} &= g_{i,t}dt + \sigma_{i,t}dW_t^Q \\ d\chi_{t,T} &= \{\sum_{i=0}^{m}\left\{g_{i,t}\cdot A_i(T-t) - \chi_{i,t}\cdot A_{i-1}(T-t)\right\}\}dt + \sigma_{t,T}dW_t^Q\end{aligned} \tag{19}$$

g is the difference (by Girsanov's theorem) between the observed drift and the market price of the risk λ_t multiplied by the volatility, $g_i = \mu_{i,t} - \sigma_{i,t}\lambda$. In this case λ is unique as long as the residual drift and volatility are polynomials of order $d-1$, and $d>m$.

Remark: *The volatility of the residual, $\sigma_{0,t,T}$ may very well be zero, which from empirical evidence, as shown in the previous section, seems to hold. Note that the volatilities remain the same under both the historical and the risk neutral measures. The volatility would be zero where the factors completely explain the futures volatility, and this is* *practical for derivative pricing.*

3.2 Fitting of Volatilities and Correlations

There is interest in modelling not only the term structure of futures prices, but also the term structure of volatilities and correlations. Each model considered in previous articles (see Schwartz 1997, 1998, Urich 2000) has different implications not only for the term structure of futures prices but also for the term structure of volatilities of futures prices and term structure of correlations of futures prices.

Proposition: *The instantaneous volatility of the futures prices* $F_{t,T}$ *, under the assumptions of Theorem 1, is:*

$$\begin{aligned} V[F_{t,T}] =& \| \sigma_{3,t} \|^2 \cdot (T-t)^4 + \| \sigma_{2,t} \|^2 \cdot (T-t)^2 + \| \sigma_{0,t,T} \|^2 + \| \sigma_{1,t} \|^2 \\ &+2\sigma_{0,t,T} \cdot \sigma_{3,t} \cdot (T-t)^2 + 2\sigma_{2,t} \cdot \sigma_{3,t} \cdot (T-t)^3 \\ &+2\sigma_{0,t,T} \cdot \sigma_{2,t} \cdot (T-t) + 2\sigma_{1,t} \cdot \sigma_{3,t} \cdot (T-t)^2 \\ &+2\sigma_{1,t} \cdot \sigma_{2,t} \cdot (T-t) + 2\sigma_{0,t,T} \cdot \sigma_{1,t} \end{aligned} \tag{20}$$

Proof: This follows from computing the variance of a linear combination of random variables in equation (1). (Note that the instantaneous volatility was denoted by $\sigma_F(t,T)$).

In the case of constant volatilities, this would be a fourth degree polynomial, which should provide a good fit to the volatility term structure of futures prices; the correlations between the factors will determine whether this curve is increasing or decreasing. In practice, a rich pattern characterizes most commodities (see Schwartz 1998 for decreasing behaviour and, most recently, Gorton and Rouwenhorst 2006 for other shapes).

It is also interesting to check whether this model captures the term structure of correlations (which is decreasing in time-to-maturity for most commodities, see Schwartz 1997).

Proposition: *The instantaneous correlations of the future price* $F_{t,T}$ *at time* t*, maturing at times* T *and* T_1*, under the assumptions of Theorem 1, are provided by the expression:*

$$Corr[F_{t,T}, F_{t,T_1}] = \frac{Cov[F_{t,T}, F_{t,T_1}]}{V[F_{t,T}], V[F_{t,T_1}]^{\frac{1}{2}}} \tag{21}$$

where the covariance is given by:

$$\begin{aligned} Cov[F_{t,T}, F_{t,T_1}] =& \| \sigma_{0,t,T} \|^2 + \| \sigma_{1,t} \|^2 + (T-t)\left(\sigma_{0,t,T}\sigma_{2,t} + \sigma_{1,t}\sigma_{2,t}\right) \\ &+\sigma_{0,t,T} \cdot \sigma_{1,t} + (T_1-t)\left(\sigma_{0,t,T}\sigma_{2,t} + \sigma_{1,t}\sigma_{2,t}\right) + (T-t)(T_1-t) \| \sigma_{2,t} \|^2 \\ &+(T-t)^2\left(\sigma_{0,t,T}\sigma_{3,t} + \sigma_{1,t}\sigma_{3,t}\right) + (T_1-t)^2\left(\sigma_{0,t,T}\sigma_{3,t} + \sigma_{1,t}\sigma_{3,t}\right) \\ &+\left[(T-t)(T_1-t)^2 + (T-t)^2(T_1-t)\right]\sigma_{2,t} \cdot \sigma_{3,t} + (T-t)^2(T_1-t)^2 \| \sigma_{3,t} \|^2 \end{aligned} \tag{22}$$

Proof: The expression for the covariance follows from using the model and the fact that covariance is a bilinear function.

This is a quotient of polynomials in two variables of order 4 (for constant volatilities); the correlation on the factors will determine whether this structure is increasing or decreasing.

Proposition: *In the setting of Theorem 1, the distribution of* $F_{t,T}$ *, given a* σ *-algebra* F_0*, is normal under the* Q *-measure. Its mean is* $F_{0,T}$ *and its variance follows from the integration of equation (20).*

Proof: The conditional normality comes from the multivariate normality of the underlyings (see Bjork, 2002. It is known that n -dimensional linear stochastic differential equations,

$$dX_t = (A_t \cdot X_t + b_t)dt + \sigma_t dW_t \tag{23}$$

where A_t is an $n \times n$ matrix deterministic function and b and σ are R^n value deterministic function, are multivariate normal distributed. The solution of this SDE is:

$$X_t = \Psi_{t,0} \cdot x_0 + \int_0^t \Psi_{t,s} \cdot \sigma_s dW_s, \tag{24}$$

where the deterministic matrix function $\Psi_{\cdot,\cdot}$ is called the fundamental matrix of $A_\cdot$. It satisfies:

$$\begin{aligned} &\frac{\Psi_{t,s}}{dt} = A_t \cdot \Psi_{t,s} \\ &\Psi_{s,s} = I. \end{aligned} \tag{25}$$

It follows that $F_{t,T}$ is a linear combination of normal distributions.

Now, the expectation and covariance are computed. From the martingale property of futures prices under the Q -measure, it follows that:

$$\begin{aligned} E^Q\left[\left(\chi_{t,T}, S_t, \xi_t, \eta_t\right)/0\right] &= \Psi_{t,0}\cdot\left(\chi_{0,T}, S_0, \xi_0, \eta_0\right) \\ Cov^Q\left[\left(\chi_{t,T}, S_t, \xi_t, \eta_t\right)/0\right] &= \int_0^t \Psi_{t,s}\cdot\sigma_s\cdot\sigma_s{}'\Psi_{t,s}{}'ds, \end{aligned} \tag{26}$$

4. Applications: Commodities Derivatives

The normality of our model is beneficial in the pricing of well-known derivatives. It is not difficult to show that closed-form solutions can be obtained for several well-known derivatives, such as a European Option on a futures contract and an Arithmetic Asian Option on a spot or futures contract (where the payoff at t is $\left\{\int_0^t F_{u,T-u}du - K\right\}^+$), and we propose some others in Section 4.1

The objective of this section is the proposal of new derivatives whose underlyings represent mathematical features of the term structure. Moreover, a new family of derivatives, that can be called conditional derivatives, is presented, where the payoff conditions a set of random variables on the value of a second correlated set of random variables.

In order to find the price at time zero of a contingent claim on a future contract starting at t with maturity T, given information at a given initial point 0, the conditional expectation and volatility under the Q-measure are needed. This was provided in the propositions

4.1 Derivatives Based on the Proposed Model

In this section several derivatives are proposed that aim to protect against and/or take advantage of contango (Co) and backwardation (Ba) movements. These derivatives will be based on the new underlyings created, and will keep track of those movements.

Basically, a payoff is created, using spot or futures prices, conditioning on the sign or the magnitude of the factors (slope and curvature) of the new underlyings. The reasoning is that a company may want protection against some inconvenient movements of the term structure curve while it benefits from other convenient movements. The idea of creating derivatives respect to the underlyings may be extended to other models (see Schwartz 1997). But there are two problems with this extension: first, the meaning of the underlyings may not be clear (which affects the usefulness of the derivatives); and secondly these derivatives can be constructed by using already available derivatives from futures and bonds, so there is no need to create them.

The derivatives presented here could be traded as long as a market for the underlying has been created, the reason being that there is no unique method of estimating the current value of these underlyings (in particular, their slope and curvature) and therefore the market should have the last word. Such a market could be seen as complementary to and compatible with the existing futures market. Note that, in particular, the market for spot prices was originally created using short-term futures prices until enough trades made it liquid.

A few examples, which are by no means exhaustive, are set out below.

1) A contract with a maturity date T, which gives the right to receive a future contract, with maturity $T_1 > T$, if the market is in contango at T: $\xi(T) > 0$.

Payoff: $F(T,T_1)\cdot 1_{\xi(T)>0}$.

Note that $\xi(T) > 0$ is equivalent to $\frac{\partial F_{t,T}}{\partial T}\big|_{T=t} > 0$ which intuitively could be approximated by $F(T,T_1) > F(T,T)$ or more precisely $\sum_{i=1}^{n}\frac{F_{t,T_i} - F_{t,T_{i-1}}}{(n-1)(T_i - T_{i-1})}$.

Working with ξ is more meaningful as it summarizes the trend of the term structure; for instance, an investor may not want to receive a future contract even if $F(T,T_1) > F(T,T)$ as this could be an isolated event, ($F(T,T_2) < F(T,T)$ for other T_2). Therefore this derivative protects the owner against changes in the whole curve from Co to Ba.

2) Option, maturity date T, which gives the right to receive a future contract, with maturity $T_1 > T$, if the term structure decreases ($\xi > 0$), or the spot price at T if ($\xi < 0$).

3)

$$\begin{cases} F_{T,T_1} & \xi_T > 0 \\ S_T & \xi_T \le 0 \end{cases} \neq (S_T - F_{T,T_1})^+$$

$$O_t = E_t^Q\left[e^{-r\cdot(T-t)}\cdot\left\{F_{T,T_1}\cdot 1_{\{\xi(T)<0\}} + S_T\cdot 1_{\{\xi(T)\ge 0\}}\right\}\right]$$

4) This derivative is not the same as $(S_T - F_{T,T_1})^+$, the main reason being that S_T could be smaller than F_{T,T_1} but the slope of the term structure could be positive ($S_T > F_{T,T_*}$, for most T_*). In other words, it is better to get involved in spot prices rather than futures prices as futures are expected to perform worse.

5) The following derivative protects against a concave downward curve. Note that a very similar derivative can be created by conditioning onto ξ instead of η (protecting against a decreasing period on the curve).

6)

$$\begin{cases} F_{T,T_1} & \eta_T > 0 \\ S_T + \xi_T\cdot(T_1 - T) & \eta_T \le 0 \end{cases}$$

$$O_t = E_t^Q\left[e^{-r\cdot(T-t)}\cdot\left\{F(T,T_1)\cdot 1_{\{\eta(T)<0\}} + \left[S_T + \xi_T\cdot(T_1 - T)\right]\cdot 1_{\{\eta(T)\ge 0\}}\right\}\right]$$

7) The term $S_T + \xi_T\cdot(T_1 - T)$ is greater than F_{T,T_1} if $\eta_T < 0$, so one is avoiding negative concavity.

8) Lookback Option (payoff):

9) $\left(\max_{u\le T}\{S_u\} + \max_{u\le T}\{\xi_u\}\cdot(T_1 - u) + \max_{u\le T}\{\eta_u\}\cdot(T_1 - u)^2 - F_{T,T_1}\right)^+$.

10) This chooses between the future price and the value of a future at the time of its maximum for spot, slope and curvature. Note that $\max_{u\le T}\{S_u\} + \max_{u\le T}\{\xi_u\}\cdot(T_1 - u) + \max_{u\le T}\{\eta_u\}\cdot(T_1 - u)^2$ is different from $\max_{u\le T}\left\{F_{u,T_1}\right\}$.

11) Lookback Conditional Option (payoff):

12)

$$\begin{cases} S_T + (T_1 - T)\cdot\max_{u\le T}\{\xi_u\} + (T_1 - T)^2\cdot\eta_T & \max_{u\le T}\{\xi_u\} > a \\ S_T + (T_1 - T)\cdot a + (T_1 - T)^2\cdot\eta_T & \max_{u\le T}\{\xi_u\} \le a \end{cases}$$

13) This considers the best slope as long as it is higher than some pre-specified value a in a period, regardless of the curvature or spot.

14) American slope (contango) Option: this gives the right to buy an underlying future price (with fixed maturity day) as soon as ξ (or η, or both) becomes negative.

This sequence can be enlarged depending on the particular needs of a company. It is intended as a starting point.

5. Conclusions

This paper develops a factor model for the term structure of commodities whose drivers could be tailor-made from a regression style analysis. For ease of presentation, the most natural drivers are selected; those corresponding to the spot, slope and curvature, $1, T-t, (T-t)^2$ respectively. These drivers lead to a set of factors with a clear meaning (derivatives with respect to time-to-maturity) and simple behaviour. The methodology is very general, allowing for any set of drivers. The goodness of fit under the historical measure shows a high R^2 value, confirming the validity of the selected drivers. The implications of a change of measure on the factors are also studied, together with expressions for the term structure of volatility and covariance. The latter is useful for derivative-pricing purposes, while the goodness of fit is relevant for risk management problems. A set of new derivatives, which go beyond the well-known spot price by adding products on the slope and the concavity, and their benefits, is provided.

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Table 1. Polynomial Term Structure Fitting.

	Mean	Min	0.01 percentile	0.05 percentile	0.95 percentile
QPol	0.9750	0.3772	0.7156	0.8858	0.999