Subordinated Binomial Option Pricing with Stochastic Arrival Intensity and Untraded Underlying Asset

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Received: April 15, 2017 Accepted: May 9, 2017 Online Published: May 15, 2017

doi:10.5430/afr.v6n2p190 URL: https://doi.org/10.5430/afr.v6n2p190

Abstract

We extend the subordinated binomial option pricing model with stochastic arrival intensity (Chang, Chang and Lu, 2010) to allow for untraded underlying assets by using matching futures prices to imply out the underlying asset values. We empirically apply the model to VIX option pricing vis-à-vis the original model with constant arrival intensity (Chang, Chang and Tian, 2006) using a two-year set of daily VIX options and futures data to specifically examine the efficacy of adding stochastic arrival intensity and untraded underlying assets. We find that the extended version significantly outperforms the original model both in sample and out-of-sample in terms of the MSE, with pricing error reduction about 37% and 32%, respectively, and additionally the outperformance is robust to the selection of the constant arrival intensity level. We attribute the outperformance to the extended model's incorporation of the stylized effects of mean-reversion and clustering in intensity arrivals as well as of the information content conveyed by the matching futures prices.

JEL Classification: G13

Keywords: Option pricing, Subordinated binomial tree, Stochastic arrival intensity, Untraded underlying assets, VIX options

1. Introduction

The Cox, Ross, and Rubinstein (1979) (CRR hereafter) binomial option pricing model and its extensions have played a central role in derivatives pricing, and are widely applied by practitioners. Chang, Chang, and Tian (2006) (CCT hereafter) have developed an extension by subordinating the CRR binomial tree to random trade arrival as the discrete-time counterpart to Chang, Chang, and Lim (1998) in the framework of the subordinated process (SP hereafter) methodology in the finance literature (see Ané and Geman (2000) for review and empirical support), to recognize that price changes are driven by news arrivals. Incorporation of random news arrival has been studied in the random-time binomial models of Dengler and Jarrow (1997), Rogers and Stapleton (1998) and Leisen (1999). Since this uncertainty is untraded and thus cannot be perfectly hedged, Rogers and Stapleton (1998) and Leisen (1999) assume that it is diversifiable, while Dengler and Jarrow (1997) hedge it by using a second option, amounting to a circular argument. To circumvent this issue, CCT introduce a shadow spanning security to hedge this arrival uncertainty - a one-news riskless numeraire bond with a price β that matures to \$1 upon the arrival of the next news and remains unchanged otherwise.

Chang, Chang and Lu (2010) (CCL hereafter) have furthered extended the CCT model to allow for stochastic arrival intensity, motivated by that trades tend to arrive in clusters with high arrival intensity around the time of important news announcements, and then revert back to a lower long run mean level after the release of the news. They do so by incorporating a discretized mean-reverting Ornstein-Uhlenbeck process to capture the above stated stylized effects. This extension could widen the appeal of the CCT model in actual applications, but it has not been empirically tested vis-à-vis the original model to examine the efficacy of the extension. Additionally, although the CCL model improves upon the CCT model theoretically, many index options traded on markets, such as VIX options, are without traded underlying assets, making the model inapplicable to price these options.

In order the CCL model can be further applied to options with untraded underlying assets, in this paper we extend the CCL model to allow for untraded underlying assets (the extended CCL model hereafter). The idea is to calibrate the underlying asset value by using traded matching futures prices (e.g. Muermann (2003). To evaluate the empirical efficacy, we test the extended CCL model against the CCT model using VIX options and futures data as VIX is untraded, in sample and out-of-sample. In the VIX option pricing literature, various stochastic volatility/jump models have been attempted, e.g. applications of Heston's (1993) stochastic volatility model and Merton's jump-diffusion model (1976). The extended CCL model considered here is unique in the sense it is derived in trading time that we price the VIX option from trade to trade to reflect arrival of information, while other models are derived in calendar time. For a recent discussion of extant empirical results see Lin and Chang (2010). Our intent is not to compare the performance to other extant VIX option pricing models, but rather to examine the empirical efficacy of adding the incremental features.

The remainder of the paper is organized as follows: Section 2 extends the CCL option pricing model to allow for untraded underlying asset. Section 3 applies the model to price VIX options vis-à-vis the CCT model in order to examine the empirical efficacy of adding stochastic arrival intensity and untraded underlying asset. Finally, Section 4 concludes the paper.

2. The Extended CCL Option pricing Model with Untraded Underlying Asset

CCT set up the SP process as follow, with constant-intensity news arrivals (we use news and trade arrival interchangeable in the context that trades take place upon news arrival) as the directing process and the underlying price changes as the parent process:

	uS	with probability *gh* one trade arrives and the underlying price jumps up at a gross rate *u*
/		
S -	*S*	with probability 1-*g* no trade arrives and the underlying price stays put
\		
	dS	with probability *g(1-h)* one trade arrives and the underlying price jumps down at a gross rate *d*,

where S denotes the underlying price; u and d denote the respective constant up and down gross jump sizes over the next time period; h and 1-h denote the respective up and down probabilities; g and 1-g denote the respective news arrival and no-arrival probabilities. The parent process, or the per-news arrival underlying price change process, is thus a stationary recombining binomial process.

CCL extend the CCT underlying asset price process by generalizing the CCT directing process to a mean-reverting Ornstein-Uhlenbeck process:

$$dj = \kappa_j(m_j - j)dt + \sigma_j dZ_j, \tag{1}$$

where κ_j denotes the speed of adjustment, m_j the long-run mean rate, σ_j^2 the instantaneous variance, and Z_j the standard Wiener process.

The solution of Eq. (1) for the time-varying intensity is known to be

$$j(v) = m_j + (j(t) - m_j)e^{-\kappa_j(v-t)} + \sigma_j e^{-\kappa_j v}\int_t^v e^{\kappa_j s} dZ_j(s), \qquad v \in [t, T], \tag{2}$$

where $j(t)$ denotes the current level of intensity. The expected intensity over a time period T-t is determined via integration as

$$E(j(T-t)) = m_j \times (T-t) + [j(t) - m_j]H_j(T-t), \tag{3}$$

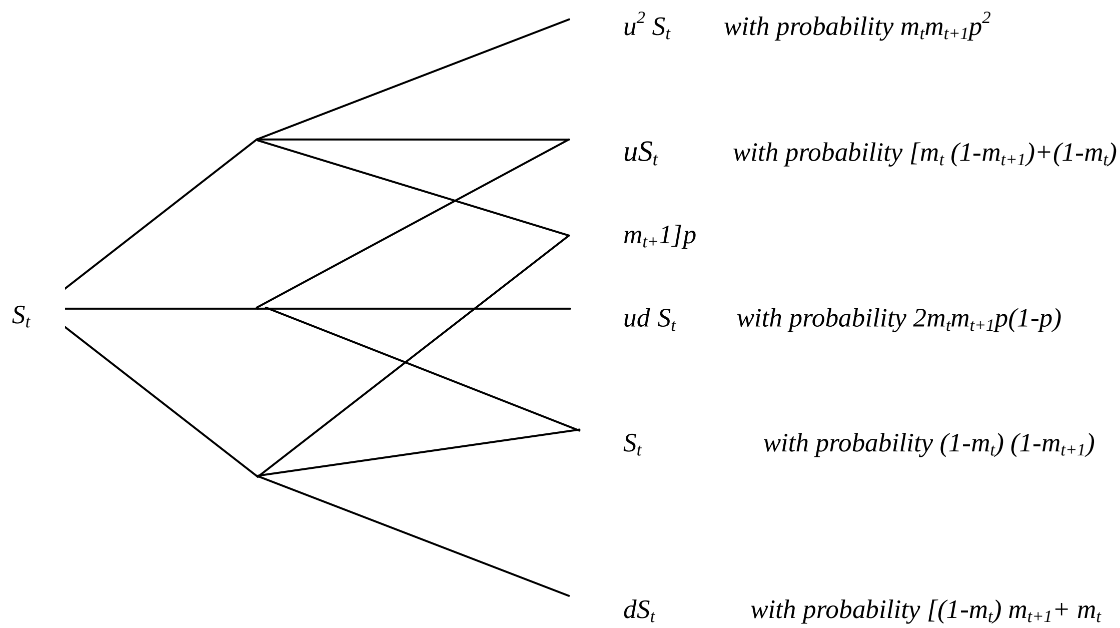
where $$H_j(T-t) = \frac{1-e^{-\kappa_j(T-t)}}{\kappa_j}.$$

With j denoting the intensity of arrival, this stochastic intensity specification incorporates a long run mean level m_j, where the speed of adjustment parameter κ_j governs the level of persistence of the intensity process – higher values of κ_j implies that the intensity process leaves the high state more quickly, and vice versa. Choosing n periods over an expected maturity of T-t to implement discretization, the probability that news (trade) will arrive in the next period is

$$g_t = E_t\left(j\left(\frac{T-t}{n}\right)\right) = \left(m_j(\frac{T-t}{n}) + [j(t) - m_j]H_j(\frac{T-t}{n})\right), \tag{4}$$

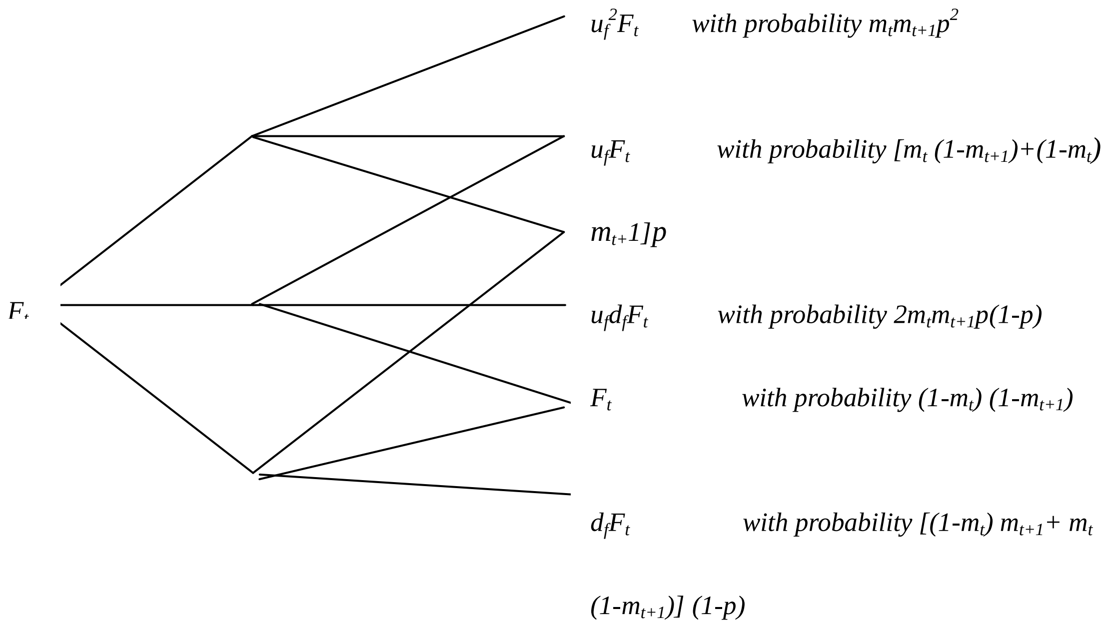
where g_t is determined by the long-run mean rate m_j, the deviation of the current level of intensity $j(t)$ from m_j, and how this deviation persists. The probability that news (trade) will arrive in the ith period where $i=2,3,\ldots,$ can be calculated as $E_t\{j(i([T-t]/n))\}- E_t\{j([i-1]([T-t]/n))\}$.

With stochastic intensity the trinomial tree is no longer stationary. We denote the respective equivalent martingale probability measures for news arrival and price change as m and p and demonstrate the tree in two time steps as illustrated below:



where news arrives with probability m_t in the first period per Eq. (4) but with probability m_{t+1} in the second period.

As the underlying asset may not be traded, we propose to imply out the underlying asset value and the corresponding price-change martingale risk measure using matching futures prices as displayed by the following two-period trinomial tree:



We derive the price-change martingale probability by calibration as in the standard CCR setup as:

$$p = \frac{1-d_f}{u_f - d_f}, \tag{5}$$

where $u_f(=exp(\sigma_f))$ and $d_f(= 1/u_f)$ are the gross up/down jump rate of the futures price upon news arrival.

As in CCL, the price of an n-period call option can be solved as a random sum of the arrival-probability-weighted normalized prices of $n+1$ k-trade-time fixed-maturity options (denoted as C_k):

$$C(n) = \sum_{k=0}^{n} M_k C_k , \tag{6}$$

where M_k is the trade-arrival martingale probability measure of k trade arrivals in n periods as illustrated above in the two-period case, and C_k is the trading-time American binomial call price with maturity k. Eq. (5) links the option value to $j(t)$, κ_j, m_j, and σ_f. The option price will increase when either the arrival probability increases in which case the tree will grow faster or when the futures volatility increases in which case the futures price jumps at a larger rate, reflecting a larger expected total price volatility. This set of parameters is sufficient to fully calibrate the underlying process and thus serves as the basis for implementing our empirical testing.

3. Empirical Tests Using VIX Options and Futures Data

We first illustrate VIX option pricing numerically to study its properties. Then we show how to implement the extended CCL model empirically through calibration by backing out the values of κ_j, m_j, and σ_f from traded VIX derivative prices in sample. Finally we empirically test the extended CCL model out-of-sample against the CCT model. We use a two-year set of daily VIX options and futures prices.

VIX is a trademarked ticker symbol for the Chicago Board Options Exchange Market Volatility Index, a popular measure of the implied volatility of S&P 500 index options. Often referred to as the fear index or the fear gauge, it represents one measure of the market's expectation of stock market volatility over the next 30 day period derived model-free based on the prices of near-term S&P 500 index options traded on CBOE. There is a formula that directly derive variances from the whole set of option prices with the same time to expiration. Two different variances for two different maturities are then interpolated or extrapolated to get the 30-day variance, leading to the VIX as the standard derivation multiplied by 100. CBOE updates the VIX values every one minutes.

Even though the VIX is quoted as a percentage rather than a dollar amount and thus is untraded, there are a number of VIX-based derivative instruments in existence, including VIX futures contracts, which began trading in 2004; exchange-listed VIX options, which began trading in February 2006; and VIX futures based exchange-traded notes and exchange-traded funds, such as S&P 500 VIX Short-Term Futures ETN (NYSE: VXX) and S&P 500 VIX Mid-Term Futures ETN (NYSE: VXZ) launched by Barclays iPath in February 2009.

In this paper we price VIX options using the extended CCL model vis-à-vis the CCT model to examine the efficacy of adding stochastic trade intensity arrival and untraded underlying asset. Our intent is not to compare the performance to other extant VIX option pricing models, but rather to examine the empirical efficacy of adding these incremental features. In the VIX option pricing literature, various stochastic volatility/jump models have been attempted, e.g. applications of Heston's (1993) stochastic volatility model and Merton's jump-diffusion model (1976). The extended CCL model considered here is unique in the sense it is derived in trading time while other models are derived in calendar time. For a recent discussion of extant empirical results see Lin and Chang (2010).

To illustrate using the extended CCT model to price VIX options numerically we consider a VIX option contract with 120 days to expiration or expected maturity at $T\text{-}t=1/3$, with the corresponding VIX futures price at value $F_t=12$. Conditional on T-t, and supposing the number of traded events that impact the VIX is no more than 40 per quarter, we set $n=40$, *i.e.* each period is three calendar days. The random arrivals of trades during $[t,T]$ imply that total number of trades in this forward period is $k€[0,40]$. To implement Eq. (13), we employ N_k time-steps to compute the binomial trade-time American futures call option prices C_k with maturity k where $k=1,2,\ldots,40$ trades. We choose $N_k = n$. N_k in principle should be as large as is computationally feasible, and as N_k increases, the binomial trees under trade maturity should converge to their counterparts under lognormal diffusion. For computational tractability, we demonstrate the methods here using $N_k=n=40$ for all k. The trade-time volatility is fixed at $\sigma_f=0.3\sqrt{k/N}$ for low volatility and $\sigma_1=0.5\sqrt{k/N}$ for high volatility. The strike prices are set at $K=10$ (in-the-money), K=12(at-the-money), and $K=14$ (out-of-the-money). An annual risk-free rate of 2% is assumed. The price results for different sets of parameterizations regarding $j(t)$, κ_j, m_j, and σ_f are shown in Table 1 below.

Table 1. VIX Option Pricing

Prices in VIX value	σ_1=0.3			σ_1=0.5		
Different parameterizations	K=10	K=12	K=14	K=10	K=12	K=14
m_j =80, κ_j =15, $j(t)$ =100	4.73	3.83	2.16	5.99	4.78	3.54
m_j =80, κ_j =30, $j(t$ =100	4.71	3.75	2.03	5.96	4.74	3.50
m_j =80, κ_j =2, $j(t)$ =100	4.81	3.94	2.13	6.17	4.95	3.71
m_j =80, κ_j =15, $j(t)$=60	4.66	3.70	1.72	5.85	4.63	3.35
Constant intensity $m_j\Delta t$ =0.6667	4.70	3.75	1.75	5.92	4.76	3.36

VIX options are priced based on expected maturity of T=0.333, current futures price of F_t=12, discretization scheme of N=n=40, and different sets of parameterizations schemes regarding $j(t)$, κ_j, m_j, and σ_1, where $j(t)$ denotes the current intensity level, κ_j the speed of adjustment, m_j the long-run mean intensity rate, and σ_j^2 the instantaneous variance.

From Table 1, it can be seen that the European-styled VIX option prices increase significantly with increase in trade-time volatility σ_1, with moneyness, and with decrease in κ_j, as $j(t)\text{-}m_j=20$ here indicates a downward adjustment of the current intensity level $j(t)$ toward the long-run mean m_j. Comparing with the case of long-run constant intensity by setting $j(t)=m_j=80$, it is seen that whenever $j(t)>m_j$, the option prices will be higher than in the case of constant intensity.

To examine empirically the efficacy of the extension, we use the daily data on VIX options and futures, from Aug. 01, 2006 to July 31, 2008, which we obtain from the Chicago Board Options Exchange. We consider the next three maturities, five near-the-money strike prices, and both calls and puts. To avoid the bid-ask bounce problem, we concentrate on the midpoint bid-ask quote instead of actual transaction prices. To clean up the option data we apply standard data filters. Since the VIX options are European, we remove the present value of all dividends prior to the maturity of the option from the index value before we use an option pricing model. We obtain daily dividend series from Standard and Poor's Corporation through the DRI database, and daily T-bill yields from the *Wall Street Journal*.

The final sample consists of 15,180 option prices and 3,030 futures prices over 506 trading days. A detailed description of the sample is available on request.

In Table 2, we report parameter estimates from both CCT's constant intensity model and the extended CCL model. In the former there are two estimates while in the latter there are five estimates. The estimation period is from Aug. 01, 2006 to July 31, 2007. Both models are fitted to VIX option prices by minimizing mean squared pricing errors (MSE). Grid search is first used to find an appropriate initial estimate. A nonlinear optimization routine (Matlab: fmincon) is then used to find the optimal estimates. The results clearly show the extended CCL version uniformly fits better the data than the constant intensity counterpart based upon the MSE criterion, with the reduction of pricing error about 37%. In addition the outperformance is robust to parameter estimates that for every constant intensity (or long-term mean intensity in the case of stochastic intensity) estimate in the set of [49, 100, 200], the extended model outperforms the constant intensity counterpart by a wide margin. In order to resolve the overfitting issue, next we test the two versions out-of-sample.

Table 2. In-Sample Parameter estimation

Panel A: Constant intensity and traded underlying assets									
m_J	49	100	200						
σ_1	1.080	0.755	0.755						
MSE	0.430	0.430	0.430						
Panel B: Stochastic intensity and untraded underlying assets									
m_J	49	49	49	100	100	100	200	200	200
σ_1	1.080	1.080	1.080	0.755	0.755	0.755	0.755	0.755	0.755
J_0	49	44.1	53.9	100	90	110	200	180	220
κ_j	0	0	0	1.5	1.6	1.2	0.944	0.969	0.920
σ_J	300	300	300	96.67	96.67	80	120	120	120
MSE	0.289	0.286	0.293	0.293	0.292	0.293	0.288	0.288	0.288

In this table, we report parameter estimates from both the CCT constant intensity model and the extended CCL stochastic intensity with untraded underling assets model. For stochastic intensity, J_0 denotes the current intensity level, κ_j the speed of adjustment, m_j the long-run mean intensity rate, and σ_j^2 the instantaneous variance. The estimation period is from Aug. 01, 2006 to July 31, 2007. Both models are fitted to VIX option prices by minimizing mean squared pricing errors. Grid search is first used to find an appropriate initial estimate. A nonlinear optimization routine (Matlab: fmincon) is then used to find the optimal estimates.

Next, in Table 3, we report out-of-sample fit of both models of VIX option prices. Model parameters are taken from the in-sample estimates in the period from Aug. 01, 2006 to July 31, 2007 (as reported in Table 2). The estimated model is then fitted out of sample in the period from Aug. 01, 2007 to July 31, 2008. Mean squared pricing errors are reported for both models. The results clearly reconfirm the merits of the stochastic intensity and untraded underlying asset extension that with the extension the MSE is significantly reduced from 0.626 to 0.426, i.e. about a 32% reduction in pricing error. In addition the outperformance is robust to parameter estimates that for every constant intensity (or long-term mean intensity in the case of stochastic intensity) estimate in the set of [49, 100, 200], the extended model outperforms the constant intensity counterpart by a wide margin. We attribute the outperformance to the extended model's incorporation of the stylized effects of mean-reversion and clustering in intensity arrivals as well as of the information content conveyed by the matching futures prices.

Table 3. Out-of-sample tests

Panel A: Constant intensity and traded underlying assets			
m_J	49	100	200
σ_1	1.080	0.755	0.755
MSE	0.626	0.626	0.626

Panel B: Stochastic intensity and untraded underlying assets									
m_J	49	49	49	100	100	100	200	200	200
σ_1	1.080	1.080	1.080	0.755	0.755	0.755	0.755	0.755	0.755
J_0	49	44.1	53.9	100	90	110	200	180	220
κ_j	0	0	0	1.5	1.6	1.2	0.944	0.969	0.920
σ_J	300	300	300	96.67	96.67	80	120	120	120
MSE	0.543	0.536	0.549	0.462	0.462	0.462	0.467	0.471	0.464

In this table, we report out-of-sample fit from both the CCT constant intensity model and the extended CCL stochastic intensity with untraded underling assets model to VIX option prices. Model parameters are taken from the in-sample estimates in the period from Aug. 01, 2006 to July 31, 2007 (as reported in Table 2). For stochastic intensity, J_0 denotes the current intensity level, κ_j the speed of adjustment, m_j the long-run mean intensity rate, and σ_j^2 the instantaneous variance. The estimated model is then fitted out of sample in the period from Aug. 01, 2007 to July 31, 2008. Mean squared pricing errors are reported for both models.

4. Concluding Remarks and Futures Research Direction

We extend the CCL subordinated binomial option pricing model with stochastic arrival intensity to allow for untraded underlying assets by using matching futures prices to imply out the underlying asset values. As in CCL, the price of an *n*-period call option can be solved as a random sum of the arrival-probability-weighted normalized prices of *n+1 k*-trade-time fixed-maturity options, but unlike in CCL the price-change martingale probability measure now depends on the gross up/down jump rate of the futures price upon news arrival. The option price will increase when either the arrival probability increases in which case the tree will grow faster or when the per jump futures volatility increases in which case the futures price jumps at a higher rate upon news arrival, reflecting a larger expected total price volatility. This new pricing formula is consistent in the context of Lin and Chang (2010) in that the option and the futures price are priced consistently.

We empirically apply the model to VIX option pricing vis-à-vis the original subordinated option pricing model of CCT to examine the empirical efficacy of the added incremental features using a two-year set of daily VIX options and futures data. We find that the extended version significantly outperforms the original model both in sample and out-of-sample in terms of the MSE, with pricing error reduction about 37% and 32%, respectively, and the outperformance is robust to the selection of the constant arrival intensity level. We attribute the outperformance to the extended model's incorporation of the stylized effects of mean-reversion and clustering in intensity arrivals as well as of the information content conveyed by the matching futures prices.

In order to further examine the efficacy of the extended CCL model, in particular the efficacy of a time change from calendar time to trading time, we propose to extend the empirical test by benchmarking to other prevailing VIX option pricing model in a futures project.

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