Optimal Leverage of Commercial Real Estate Investments: A Trade Off Option Theoretic Approach

Ángela C. De Luna López[1], Walter L. De Luna Butz[2] & Luis Moreno Salas[2]

[1] Ernst and Young Madrid, Madrid, Spain

[2] Ibero Capital Management, Madrid, Spain

Correspondence: Walter L. De Luna Butz, Ibero Capital Management, 28006 Madrid, Spain. E-mail: wdeluna@iberocm.com

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Abstract

The determination of an optimal level of financial leverage for commercial real estate investment transactions is a central concern both for investors and financial institutions. Investors seek the maximization of the economic value of the investment firm while financial institutions need to adequately price the overall risk in accordance with the available regulatory capital budget. Excess leverage and large and/or unexpected shifts in the default risk of commercial real estate portfolios are a relevant potential threat to both objectives and need to be addressed from the perspective of macro prudential guidance. Traditional modelling of (commercial real estate) asset values as geometric Brownian motions may not be sufficient to capture tail events and risks relevant to investors and financial institutions for pricing and risk assessment purposes, since the distribution of returns of such properties does not seem to be normal. We examine the default, prepayment risk and optimal unitranche leverage level of single-borrower non- recourse mortgage loans, based on a trade-off option theoretic structural approach that captures asymmetry and kurtosis of asset values' returns distributions through the shifted lognormal distribution as in De Luna et al (2025). Our framework includes distress costs expanding the firm value paradigm of Modigliani and Miller (1958, 1963) to model interior optimal leverage solutions. We conduct several numerical experiments to examine how the different parameters of the model but also the returns distribution and load of distress costs affect the pricing of the CRE mortgage loan and the level of optimal and prudent leverage.

Keywords: real estate investments, optimal capital structure, structural credit model

1. Introduction

Commercial real estate is one of the largest asset classes available and therefore attracts substantial capital investment worldwide. Due to the size of the business, commercial real estate mortgage loans granted to all kinds of investors and firms constitute a relevant part of any western commercial banking institution. The mortgage system is designed to, in principle, protect the priority interest of the lender in case of a default event occurring. Relatively high loan to value lending is a longstanding not uncommon practice under the perception that commercial real estate collateral offers high recovery of debts outstanding. On the other hand, episodes like the 2008 financial crisis have evidenced that large losses may occasionally occur, exposing large portfolios and potentially threatening the stability of financial institutions. Lending guidelines and standards too often focus on general LTV levels and income production of the mortgaged property, especially for non-recourse loans, while specific exposure characteristics like the dynamic behavior of the type of property, its specific location and cost of applicable legal system are ignored, forfaiting critical determinants of default and recovery expectancies of the individual exposure. De Luna et al (2025) develop a pricing model based on a shifted lognormal distribution of the residential collateral's value returns. The shifted lognormal distribution takes a better account of extreme drops in value events than traditional modelling based on GBMs. It arrives at higher credit spreads for a given LTV than in conventional models and shows that the collaterals idiosyncratic return distribution plays a crucial role in correctly assessing credit and prepayment risk. But how much leverage is too much for a given proposed commercial real estate investment project? Any leverage beyond an optimal amount (the optimal capital structure) for a single investment project shall be deemed excessive and does not respond to a firm or equity value maximization objective, but rather to other arguments like shifting the target risk return profile along the capital market line, capital rationing or risk separation strategies. The determination of such optimal capital structure shall increase the lender's awareness about the arguments underlying the sponsor's design of the transaction and the appropriateness of such risk taking. For the purposes of this study, optimal capital structure shall be referred to the combination of equity and debt that maximizes the project (firm) value (Note 1).

The capital structure decision is critical for any corporation or investor. Any acceptable investment should offer returns exceeding its cost of capital (Modigliani and Miller (1958) Proposition III) and such cost of capital can only be evaluated based on a specific mix of debt and equity.

The existence of an optimal mix of debt and equity contravenes the strict sense of the financing policy irrelevance theorem of Modigliani and Miller (MM (1958)) but only because MM (1958) established their famous Proposition I based on the restrictive assumptions of perfect capital markets and absence of transaction costs which include taxes and bankruptcy costs (Note 2). The relationship between the unlevered and the levered firm value was later restated in MM (1963) to include the present value of interest tax shields such that,

$$VL = Vu + PVTS \text{ (Note 3)} \tag{0}$$

Robicheck and Myers (1966), Baxter (1967) and Kraus and Litzenberger (1973) have shown that in the presence of bankruptcy costs and assuming the independence of Vu from the debt level, the restated Proposition I of equation 0 can be redefined as in equation (1) (Note 4):

$$VL = Vu + PVTS - PVBC \text{ (Note 5)} = D + E \tag{1}$$

In what follows, we shall conceptualize the valuation equation of the CRE investment in accordance with equation (1). By specifically defining bankruptcy costs (BC) and interest tax shields (ITS or TS) dependent on default/solvent states we will locate the optimal point of leverage where marginally PVTS is still larger than PVBC. Such is the framework proposed by the so- called Static Trade Theory of Capital Structure which defends that firm leverage is determined by balancing interest tax shields against any kind of bankruptcy cost. (Note 6) Our model is inspired by contingent claims valuation framework of Black and Scholes (1973) and Merton (1974) and adapts and extends the pricing model of De Luna, A. et al (2025) in several direction to arrive at an endogenous optimal capital structure for the project analyzed.

In section 2, we perform numerical experiments under plausible parameter values and discuss the implications of (i) a shifted log-normal distribution of the collateral assets' returns under several parameter sets (ii) the interaction of interest tax shield and bankruptcy costs on project value under alternative definitions (iii) the location of the optimal mix of the market value of debt and equity (iv) implications for lending standards best practices. In section 3, we discuss in detail our methodology and its limitations and conclude in section 4. Appendix A includes the R code of the model, the parameters and all the results.

2. Results and Discussion

According to the Static Trade Off Theory (STOT) (Note 7), the optimal amount of debt (OCS) is the one that maximizes the net tax shield (NTS), that is, the present value of interest tax shield (PVTS) minus the present value of bankruptcy costs (PVBC). According to equation 1, maximizing the NTS maximizes the levered firm value. At inception and to avoid arbitrage, debt needs to be priced to par (where the market value of debt is equal to its face value) i.e., the credit risk premium (the option adjusted spread (OAS)) covers all relevant risks, most notably, the default and prepayment risk, as detailed in De Luna, A.C. et al (2025). Locating the optimal capital structure requires therefore searching for the resulting levered firm value through marginally increasing and repricing the total amount of debt, the interest tax shields and the present value of bankruptcy costs. Finding the OCS implies consequently also correctly pricing the debt. At each amount of debt, the PVTS will be evaluated under the assumption that all interest tax shield is lost upon default and reorganization of the firm. The equilibrium interest results from adding the OAS to the risk- free rate such that at inception, the face value of debt equals its market value. Equally, bankruptcy costs only arise in case of default. Simplifying, and only for the purpose of conceptual guidance, equation (1) may be restated as:

$$VL = Vu + ps\ PVTS + (1-ps)\ PVBC = E + D \tag{2}$$

Where ps is ps (i,t), the accumulated survival probability in any state and time.

The problem of the optimal capital structure based on a structural option theoretic model has been analyzed by Scott (1977), Brennan and Schwartz (1978), Leland (1994), Leland and Toft (1996) and many others but most often, based on modelling the asset value (the collateral) as a geometric Brownian motion. Since we are specifically analyzing non-recourse, unitranche mortgage financing of an income producing (commercial) real estate asset, the risk characteristics of the debt heavily depend on the risk-return profile of the asset financed (the collateral (Note 8)). The purpose of our analysis is to determine how more realistic distributional assumptions of the collaterals' value affect not only the risk of debt (as in A.C. De Luna et al (2025) for residential mortgages) but also the level of its optimal amount, that is, the debt level that maximizes the levered firm value.

From the viewpoint of the lender, determining the optimal capital structure of the investment to lend against has at least one important advantage: the LTV requested by the investor can be assessed not only in absolute terms but in relation to the optimal LTV of a given asset. A 70% LTV transaction might be demanding for assets where the optimal debt is 50%, while it might be a "core" type of deal for an asset where the optimal debt mix is at 65%. Where precisely the optimal debt amount is located, as we shall observe in what follows, is a consequence of (i) the distributional characteristics of the asset analyzed (ii) the structural elements of the transaction itself (repayment schedule inter alia) and (iii) structural elements of the economic environment (bankruptcy costs and procedure and limits to the tax deductibility of interest. From the lender's and even the regulator or supervisor perspective, the riskiness of the loan and the collateral is now not based on experience or "expert judgement", but a quantitatively supported conclusion.

The optimal capital structure, when determined ex ante, should be based on pricing to par of the debt, i.e. market value of the debt is equal to its face value. This is consistent with a no- arbitrage assumption.

The valuation strategy to obtain the optimal capital structure is (i) start with a reasonable amount of debt and price it to par (ii) value the interest tax shield and the bankruptcy costs (iii) derive the levered firm value (iv) repeat over a enough number of debts amounts (different LTV's) to locate de highest resulting firm value.

We will follow De Luna A.C. et al in applying the shifted lognormal binomial tree of Haahtela (2006) to the valuation model of the debt with an important addition: the rational default trigger is not only that the value at time t of V(t,i), the collateral and asset to be acquired, is lower than MB(t), the mortgage balance at time t, but also, that the asset is not producing enough income (μ V) to cover the debt service obligation (DSO) (Note 9). This addition makes sense for non-recourse financing of income producing real estate since pure lack of liquidity, i.e. $\mu V(t) < DSO(t)$, but in presence of sufficient equity value, i.e., $V(t) > MB(t)$ would, in liquid markets, allow the borrower to raise equity to cover the DSO (t) payments. At maturity, the market value of debt is, as in De Luna A.C. et al (2025), equation (1):

$$MV(T,i) = \begin{cases} Max(0; V(T,i)(1-BC) & if\ MP(T) > V(T,i) \\ MP(T) & otherwise \end{cases} \quad (3)\ (\text{Note } 10)$$

Where MV(T,i) is the mortgage value at time T (maturity) and state i, MP(T) is the mortgage payment due at time T, BC are the bankruptcy costs and V(T,i) is the value of the collateral property at time T (maturity) and state i

Rolling back the tree, in the event of solvency and continuity, and considering that the mortgage loan is prepayable (Note 11), the value of the mortgage loan at any time $t < T$, is:

$$MV(t,i) = Min\left[PPV(t); \frac{pMV(t+1;i+1)+(1-p)MV(t+1;i)}{e^{rt}} + MP(t)\right] \quad (4)$$

Where MP(t) is the mortgage loan payment on t. Since we shall consider amortizing loans, MP(t) = interest (risk free rate + OAS) + interim amortization (Note 12)

Including now the default conditions:

$$MV(t,i) = \begin{cases} Max(0; V(t,i)(1-BC)(1-q) & if\ MV(t,i) > V(t,i)\ and\ \mu V(t,i) < DSO(t) \\ Min[PPV(t); CV(t) + MP(t)](1-q) + qPOS(t)RR & otherwise \\ CV(t) = \frac{pMV(t+1;i)+(1-p)MV(t+1;i+1)}{e^{rt}} & \end{cases} \quad (5)$$

Where POS(t) is the principal amount outstanding on t, RR is the constant recovery rate of such principal amount upon suboptimal default event, CV(t) is the continuation value of the mortgage loan, q the constant risk neutral probability of a suboptimal default event and p the risk neutral probability of an up move in the binomial tree. Since MP(t) includes interest payments = (r + OAS) x MB(t-1), it is possible to iterate to find the par (fair) OAS such that,

$$MV(0;0) = N$$

where OAS is the option adjusted spread and N is the nominal or face value of the debt.

Next, we need to define the valuation equations of the interest tax shields to arrive at the PVTS as required by equation (2). At maturity:

$$ITS(T,i) = \begin{cases} 0 & if\ MP(T) > V(T,i) \\ Min(iT; L\mu V(t,i))(1-q) & otherwise \end{cases} \tag{6}$$

Where μ = EBiT is the earnings before interest and taxes of the vehicle investing in the property, as a percentage of the collateral's value V(t,i) and L is the legal limit, as a percentage of EBIT of deductible interest expense. In our context EBIT = μ.

Rolling back the interest tax shield tree:

$$MV(t,i) = \begin{cases} 0 & if\ MV(t,i) > V(t,i)\ and\ \mu V(t,i) < DSO(t) \\ Min[PPV(t); CV(t) + Min(i(t)T; L\mu V(t,i))](1-q) & otherwise \\ CV(t) = \frac{pITS(t+1;i)+(1-p)ITS(t+1;i+1)}{e^{rt}} & \end{cases} \tag{7}$$

Equation 7 states that the interest tax shield is lost in case of default where default occurs either because MV(t,i) > V(t,i) and μV(t,i) < DSO(t) or for other reasons with risk neutral probability q. In case of solvent continuation, ITS(t,i) has a maximum of L μV(t,i), where L is the maximum interest as a percent of EBIT that is tax deductible. PPV(t) is the prepayment amount, i.e., the amount the debtor needs to pay to early terminate the financing agreement.

The next component of the levered firm value VL that needs to be valued according to equation (2) is the PVBC. Since BC appears only when the firm is in default as part of the amount recovered by the debt investor, we shall estimate PVBC as the difference between the market value of the debt with bankruptcy costs (D^+) and the market value of debt as if BC = 0 (D^-).

In what follows we will analyze the mortgage financing agreement and the corresponding optimal debt amount that maximizes the levered firm value based on a discretization scheme following Haahtela's (2006 and 2012) shifted lognormal binomial tree which works under the up and down jump parameterization of Wilmott et al (1995). Our base case numerical experiment parts form the following set of plausible parameters of Table 1:

Table 1. Displaced diffusion model inputs for a 5- year amortizing mortgage loan

							x dt
S(0)		100.00		BC		**15.00%**	
Θ		**10.00**		q		**0.00%**	
S(Θ) = (S(t)+θ)		110.00		Rec. rate (RR) if q		**80.00%**	
σ(d)		**20.00%**		IPA pa		**2.00%**	
r = ES5		**2.25%**		Corp Tax Rate (T)		**25.0%**	
δ		**2.00%**		μ = EBIT pa		**4.00%**	1.00%
r-δ		0.25%		Tax shield limit (L)		**30.00%**	0.30%
exp(rdt)		1.006		PPF (Prepaym. fee)		**1.50%**	
T		**5**		u		1.101	
n		**20**		d		0.900	
dt = T/n		0.25		p		0.50	

Where:

S(0) is the property value at time 0

Θ is the shifting parameter of Haahtela's shifted lognormal distribution

S(θ) the shifted property value

σ(d) is the volatility of S(θ)

r = ES 5 is the 5- year euroswap (risk free) rate (Note 13)

δ is the operating costs of the property company (service charges, property tax, insurance etc)

r-δ is the risk neutral drift of the property of S(θ)

T is the maturity of the mortgage loan (5 years) (Note 14) n is the number of steps of the shifted lognormal binomial tree

$dt = T/n$ is the time step of the tree

BC are the percent bankruptcy costs

q is the risk neutral probability of a suboptimal default with recovery rate RR

IPA pa is the interim principal amortization per annum as percentage of initial loan amount

T is the corporate tax rate, L the amount of interest as percentage of EBIT which is tax deductible

μ is the EBIT of the firm and PPF the prepayment fee as a percentage of total amount prepaid

u, d and p are the Shifted Lognormal binomial tree parameters, such that (Note 15):

$$u = e^{((r-\delta)dt)}\left(1+\sqrt{e^{\sigma_d^2 dt}}\right)-1 = 1.101$$

$$d = e^{((r-\delta)dt)}\left(1-\sqrt{e^{\sigma_d^2 dt}}\right)-1 = 0.900$$

$$p = 0.5$$

Figure 1 shows the 20 steps shifted binomial tree of S(θ) tree with the inputs of Table 1:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
110.0	121.1	133.3	146.8	161.6	177.9	195.9	215.6	237.4	261.4	287.8	316.8	348.8	384.0	422.7	465.4	512.4	564.1	621.1	683.7	752.8
	99.0	109.0	120.0	132.2	145.5	160.2	176.3	194.1	213.7	235.3	259.1	285.2	314.0	345.7	380.6	419.0	461.3	507.9	559.1	615.6
		89.2	98.2	108.1	119.0	131.0	144.2	158.8	174.8	192.4	211.9	233.2	256.8	282.7	311.2	342.7	377.2	415.3	457.2	503.4
			80.3	88.4	97.3	107.1	117.9	129.8	142.9	157.4	173.3	190.7	210.0	231.2	254.5	280.2	308.5	339.6	373.9	411.7
				72.3	79.6	87.6	96.4	106.2	116.9	128.7	141.7	156.0	171.7	189.1	208.1	229.2	252.3	277.7	305.8	336.6
					65.1	71.6	78.9	86.8	95.6	105.2	115.9	127.6	140.4	154.6	170.2	187.4	206.3	227.1	250.1	275.3
						58.6	64.5	71.0	78.2	86.1	94.7	104.3	114.8	126.4	139.2	153.2	168.7	185.7	204.5	225.1
							52.7	58.1	63.9	70.4	77.5	85.3	93.9	103.4	113.8	125.3	138.0	151.9	167.2	184.1
								47.5	52.3	57.6	63.4	69.8	76.8	84.6	93.1	102.5	112.8	124.2	136.8	150.6
									42.7	47.1	51.8	57.0	62.8	69.1	76.1	83.8	92.3	101.6	111.8	123.1
										38.5	42.4	46.6	51.4	56.5	62.2	68.5	75.5	83.1	91.5	100.7
											34.7	38.1	42.0	46.2	50.9	56.0	61.7	67.9	74.8	82.3
												31.2	34.3	37.8	41.6	45.8	50.5	55.6	61.2	67.3
													28.1	30.9	34.0	37.5	41.3	45.4	50.0	55.1
														25.3	27.8	30.6	33.7	37.1	40.9	45.0
															22.8	25.1	27.6	30.4	33.4	36.8
																20.5	22.6	24.8	27.4	30.1
																	18.5	20.3	22.4	24.6
																		16.6	18.3	20.1
																			15.0	16.5
																				13.5

Figure 1. S(θ) 20 steps shifted binomial tree

A 20 steps tree is a natural choice since it implies quarterly observations or checks of the real estate price in line with standard practice. (Note 16) The base case assumption here is also that the loan provisions require quarterly interest and amortization payments, also in line with prevailing lending standards.

Figure 2 provides the S(t) values in a shifted lognormal binomial tree of 20 steps:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
100	111.0	123.2	136.6	151.4	167.6	185.5	205.2	226.9	250.9	277.2	306.2	338.1	373.2	411.9	454.5	501.4	553.1	610.0	672.6	741.6
	89.0	98.9	109.9	121.9	135.2	149.8	165.9	183.7	203.2	224.7	248.4	274.5	303.3	334.9	369.7	408.1	450.3	496.8	548.0	604.4
		79.0	88.0	97.8	108.7	120.6	133.8	148.3	164.3	181.9	201.2	222.5	246.0	271.9	300.4	331.7	366.2	404.3	446.1	492.2
			70.1	78.1	87.0	96.8	107.5	119.4	132.4	146.8	162.6	180.0	199.2	220.4	243.6	269.3	297.5	328.6	362.8	400.5
				62.0	69.3	77.3	86.0	95.7	106.4	118.1	131.0	145.3	161.0	178.2	197.3	218.2	241.3	266.7	294.7	325.5
					54.8	61.3	68.5	76.4	85.1	94.7	105.2	116.9	129.7	143.8	159.3	176.5	195.3	216.1	238.9	264.1
						48.2	54.1	60.5	67.7	75.5	84.1	93.6	104.1	115.6	128.3	142.3	157.7	174.7	193.4	213.9
							42.3	47.6	53.4	59.8	66.8	74.6	83.2	92.6	102.9	114.4	127.0	140.8	156.1	172.9
								37.0	41.8	47.0	52.7	59.1	66.0	73.7	82.2	91.5	101.8	113.1	125.6	139.4
									32.2	36.5	41.2	46.3	52.0	58.3	65.2	72.9	81.3	90.5	100.7	111.9
										27.9	31.7	36.0	40.6	45.7	51.4	57.6	64.4	72.0	80.3	89.5
											24.0	27.5	31.2	35.4	40.0	45.1	50.7	56.9	63.7	71.1
												20.5	23.6	27.0	30.7	34.9	39.5	44.5	50.0	56.1
													17.3	20.1	23.2	26.5	30.3	34.4	38.9	43.9
														14.5	17.0	19.7	22.7	26.1	29.8	33.8
															11.9	14.1	16.6	19.3	22.3	25.6
																9.6	11.6	13.8	16.2	18.9
																	7.4	9.3	11.2	13.4
																		5.5	7.2	8.9
																			3.8	5.3
																				2.3

Figure 2. S(t) 20 steps shifted binomial tree

It is noteworthy that the property value can reach values close to 0 in the worst scenarios in line with evidence drawn from experience during the financial crisis. When analyzing financial contracts where tail events are critical it is extremely important to select a correct stochastic process for the relevant variables capable of generating such empirically contrasted extreme scenarios. This issue plays a determining role in locating the optimal amount of debt and its fair price.

Figure 3 outlines the quarterly payments schedule for a 5- year mortgage loan with an initial principal of 35.00 and constant quarterly amortization of 0.5% of initial principal:

Year		0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00
Tax shield limit (L)		0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300
Principal YE	35.00	34.83	34.65	34.48	34.30	34.13	33.95	33.78	33.60	33.43	33.25	33.08	32.90	32.73	32.55	32.38	32.20	32.03	31.85	31.68	0.00
Interest		0.42	0.42	0.41	0.41	0.41	0.41	0.41	0.40	0.40	0.40	0.40	0.40	0.39	0.39	0.39	0.39	0.38	0.38	0.38	0.38
IPA		0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18	31.68
Period payment		0.59	0.59	0.59	0.59	0.58	0.58	0.58	0.58	0.58	0.57	0.57	0.57	0.57	0.57	0.56	0.56	0.56	0.56	0.56	32.05
Prepay value (PPV)		35.95	35.77	35.59	35.41	35.23	35.05	34.87	34.69	34.51	34.33	34.15	33.97	33.79	33.61	33.43	33.25	33.07	32.89	32.71	

Figure 3. Quarterly Interest and 0.5% amortization for a 5-year mortgage loan

The interest rate is the sum of the risk- free rate (in our base case the 5-year euroswap rate) and the OAS (option adjusted spread) such that the loan is priced to par, i.e. its market value is equal to its nominal or face value (Note 17). Based on equations (1) – (3), the resulting value of the 5-year mortgage loan is shown in Figure 4:

Year	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00
	35.00	35.33	35.22	35.06	34.88	34.70	34.51	34.32	34.13	33.94	33.75	33.57	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
		35.06	35.07	35.00	34.86	34.69	34.51	34.32	34.13	33.94	33.75	33.57	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
			34.67	34.77	34.75	34.65	34.50	34.32	34.13	33.94	33.75	33.57	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
				34.18	34.41	34.48	34.43	34.30	34.13	33.94	33.75	33.57	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
					33.56	33.95	34.15	34.18	34.09	33.94	33.75	33.57	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
						32.79	33.37	33.73	33.89	33.87	33.74	33.57	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
							31.80	32.62	33.20	33.52	33.62	33.54	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
								30.58	31.64	32.48	33.05	33.31	33.33	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
	Equilibrium credit spread (OAS)				0.1674%				29.11	30.39	31.53	32.40	32.92	33.09	33.00	32.81	32.62	32.43	32.24	32.06	31.87
	Interest per annum				2.4174%					27.40	28.83	30.25	31.49	32.37	32.79	32.81	32.62	32.43	32.24	32.06	31.87
											23.72	26.97	28.60	30.21	31.56	32.40	32.62	32.43	32.24	32.06	31.87
												20.41	23.33	26.55	28.45	30.33	31.79	32.43	32.24	32.06	31.87
													17.42	20.05	22.94	26.14	28.45	30.76	32.24	32.06	31.87
														14.73	17.09	19.69	22.56	25.72	28.89	32.06	31.87
															12.30	14.42	16.75	19.33	22.17	25.31	31.87
																10.10	12.00	14.10	16.42	18.97	21.79
																	8.12	9.83	11.71	13.79	16.08
																		6.33	7.86	9.55	11.42
																			4.72	6.09	7.60
																				3.26	4.49
																					1.93

Figure 4. 5-year amortizing mortgage loan value and equilibrium option adjusted spread

Figure 5 shows the ITS valuation tree following equations (6) and (7) based on the tree shown in Figure 4 and Figure 3 while full R code of the model is reflected in Appendix A:

Interest tax shield valuation																					
Year	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00
	0.941	0.95	0.91	0.86	0.81	0.77	0.72	0.67	0.62	0.57	0.53	0.48	0.43	0.38	0.33	0.29	0.24	0.19	0.14	0.10	0.05
		0.94	0.90	0.86	0.81	0.77	0.72	0.67	0.62	0.57	0.53	0.48	0.43	0.38	0.33	0.29	0.24	0.19	0.14	0.10	0.05
			0.88	0.85	0.81	0.76	0.72	0.67	0.62	0.57	0.53	0.48	0.43	0.38	0.33	0.29	0.24	0.19	0.14	0.10	0.05
				0.82	0.79	0.76	0.72	0.67	0.62	0.57	0.53	0.48	0.43	0.38	0.33	0.29	0.24	0.19	0.14	0.10	0.05
					0.74	0.74	0.71	0.67	0.62	0.57	0.53	0.48	0.43	0.38	0.33	0.29	0.24	0.19	0.14	0.10	0.05
						0.66	0.67	0.65	0.62	0.57	0.53	0.48	0.43	0.38	0.33	0.29	0.24	0.19	0.14	0.10	0.05
							0.55	0.59	0.59	0.56	0.52	0.48	0.43	0.38	0.33	0.29	0.24	0.19	0.14	0.10	0.05
								0.42	0.49	0.52	0.51	0.47	0.43	0.38	0.33	0.29	0.24	0.19	0.14	0.10	0.05
									0.24	0.37	0.44	0.44	0.42	0.38	0.33	0.29	0.24	0.19	0.14	0.10	0.05
										-	0.21	0.33	0.37	0.37	0.33	0.29	0.24	0.19	0.14	0.10	0.05
											-	-	0.19	0.28	0.30	0.28	0.24	0.19	0.14	0.10	0.05
												-	-	-	0.16	0.23	0.23	0.19	0.14	0.10	0.05
													-	-	-	-	0.13	0.17	0.14	0.10	0.05
														-	-	-	-	-	0.10	0.10	0.05
															-	-	-	-	-	-	0.05
																-	-	-	-	-	-
																	-	-	-	-	-
																		-	-	-	-
																			-	-	-
																				-	-
																					-

Figure 5. Interest tax shield valuation for the 5-year amortizing mortgage loan

The tree in Figure 4 can be recalculated with BC = 0 obtaining a new loan (market) value. The difference between the new loan value and the original loan value in euros is the estimation of the bankruptcy cost. (Note 18)

Figure 6 shows the loan valuation tree with BC = 0 of the 35u 5-year amortizing mortgage loan:

Year	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00
	35.17	35.41	35.25	35.07	34.89	34.70	34.51	34.32	34.13	33.94	33.75	33.57	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
		35.33	35.20	35.05	34.88	34.70	34.51	34.32	34.13	33.94	33.75	33.57	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
			35.08	34.98	34.84	34.68	34.50	34.32	34.13	33.94	33.75	33.57	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
				34.80	34.74	34.63	34.48	34.31	34.13	33.94	33.75	33.57	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
					34.49	34.46	34.39	34.27	34.12	33.94	33.75	33.57	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
						34.13	34.16	34.13	34.05	33.92	33.75	33.57	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
							33.73	33.80	33.84	33.81	33.71	33.56	33.38	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
								33.27	33.38	33.48	33.52	33.48	33.36	33.19	33.00	32.81	32.62	32.43	32.24	32.06	31.87
	Equilibrium credit spread (OAS)				0.1674%				32.76	32.89	33.06	33.19	33.22	33.15	33.00	32.81	32.62	32.43	32.24	32.06	31.87
	Interest per annum				2.4174%					32.23	32.34	32.55	32.77	32.92	32.93	32.81	32.62	32.43	32.24	32.06	31.87
											27.91	31.73	31.94	32.24	32.52	32.67	32.62	32.43	32.24	32.06	31.87
												24.01	27.45	31.24	31.57	31.99	32.34	32.43	32.24	32.06	31.87
													20.50	23.59	26.99	30.75	31.26	31.87	32.24	32.06	31.87
														17.33	20.10	23.16	26.54	30.26	31.11	32.06	31.87
															14.47	16.96	19.71	22.74	26.08	29.77	31.87
																11.89	14.12	16.59	19.31	22.32	25.63
																	9.55	11.56	13.78	16.22	18.92
																		7.45	9.25	11.24	13.43
																			5.55	7.16	8.95
																				3.83	5.28
																					2.28

Figure 6. 5-year amortizing mortgage loan value with no bankruptcy costs

The difference in this case is 0.17u. We can now value the levered firm with equation (1):

$$VL = Vu + PVTS - PVBC = 100 + 1.009 – 0.17 = 100.839$$

Repeating the procedure for several debt amounts, we obtain the results shown in Table 2:

Table 2. Optimal debt level with model inputs of Table 1.

D	D(BC=0)	ITS	VL	OAS
25.00	25.04	0.643	100.61	0.043%
30.00	30.10	0.789	100.69	0.103%
35.00	**35.17**	**0.941**	**100.77**	**0.167%**
40.00	40.38	1.129	100.75	0.314%
45.00	45.63	1.310	100.68	0.315%

Where D is the nominal and market value of the debt, D (BC = 0) is the market value of the debt with 0 bankruptcy costs (BC = 0), VL is the levered firm value and OAS is the debts option adjusted spread.

With the inputs of Table 1, the optimal amount of debt for a collateral with a value of 100 is 35 and the levered firm value gain from leverage is 0.77, less than 1%.

This is in stark contrast with, for example, a direct application of the Modigliani Miller (1963) equation:

$$VL = Vu + TD \tag{8}$$

Which would result in VL = 100 + 0.25 x 35 = 108.75, a close to 9% value gain.

Modigliani Miller's assumptions included (a) a constant perpetual predetermined risk- free amount of debt (b) full deductibility of interest tax shield which means (i) that there is no uncertainty around the collection of the interest tax shield and (ii) all interest benefits from the tax shield. Neither of these assumptions are very realistic since there is default risk that impacts interest tax shield collectability and, in many jurisdictions, there is a limit to the amount of interest that can be deducted for tax purposes (Note 19). The naïve application of the MM (1963) may lead to severe mis-valuation of the levered firm if deviations from the MM (1963) assumptions are not considered. The small gain from interest tax shields in our model can be traced back to the default risk of tax shields (with no recovery), time varying debt amount because of interim amortizations, a maturity of 5 years and a (potentially legal) limit to the amount of tax- deductible interest (30% of EBIT in our base case). The modest OAS should not confuse the analysis since such spread is charged on top of a prepayment fee of 1.5% as a base case, in line with customary practices in this type of lending. With a 0- prepayment fee (i.e. accounting for all risks in the OAS), we obtain the following results as shown in Table 3:

Table 3. Optimal debt level with prepayment fee = 0 and other inputs as in Table 1

D	D(BC=0)	ITS	VL	OAS
30.00	30.022	0.854	100.832	0.295%
35.00	35.040	1.059	101.019	0.470%
40.00	**40.075**	**1.326**	**101.251**	**0.764%**
45.00	45.118	1.365	101.247	1.363%
50.00	50.181	1.335	101.154	1.465%
55.00	55.283	1.314	101.031	1.465%
60.00	60.417	1.306	100.889	2.799%

Where ITS is the value of the interest tax shield. The OAS is quite high for all levels of debt. As discussed in De Luna et at (2025) this is a consequence of the shifted log-normal distribution of asset values, that places a higher probability mass on the tails of the distribution (leptokurtosis) significantly increasing the lender's risk of severe losses. This characteristic of the model is in line with empirical stylized facts as detailed in Young and Graff (1995) amongst others (Note 20).

The low optimal debt level is directly influenced by the limited amount of interest which is tax deductible. Table 4 shows the optimal debt level obtained when there is no limitation to the interest tax shield:

Table 4. Optimal debt level when L = 100% and inputs of Table 1

LTV = D	TS	VL	OAS
45.0	1.312	100.68	0.448%
50.0	1.564	100.72	0.740%
60.0	2.384	101.27	1.762%
65.0	2.805	101.42	2.530%
70.0	**3.136**	**101.46**	**3.573%**
75.0	3.254	101.16	5.605%
80.0	2.510	99.27	10.367%

Allowing for a full tax deductibility of interest leads to significantly higher optimal debt levels but with only a modest increase in levered firm value! Interest tax shield policies are therefore of critical importance from a macro prudential point of view since they can incentivize excessive risk- taking threatening in this way financial stability as confirmed by Dallari et al (2020). Table 5 compares optimal debt levels, OAS and levered firm values for L = 100% and L = 30% for further clarity on this point:

Table 5. Optimal debt levels for L = 100% and L = 30% and inputs of Table 1

L = 100%			
D	**TS**	**VL**	**OAS**
50.0	1.564	100.72	0.74%
60.0	2.384	101.27	1.76%
65.0	2.805	101.42	2.53%
70.0	**3.136**	**101.46**	**3.57%**
75.0	3.254	101.16	5.61%
80.0	2.510	99.27	10.37%
L = 30%			
D	**TS**	**VL**	**OAS**
35.00	**0.941**	**100.77**	**0.17%**
50.00	1.320	100.48	0.74%
60.00	1.283	100.17	1.76%
65.00	1.187	99.80	2.53%
70.00	1.034	99.35	3.57%
75.00	1.065	98.97	5.61%
80.00	0.792	97.55	10.37%

Inspection of Table 5 confirms that tax policy can effectively disincentivize excessive financial risk -taking by limiting the amount of interest to be shielded from taxation. Graham (2000) finds that firms use leverage too conservatively when compared with the implications of a trade-off theory and asks for further research to find hidden determinants. Fiscal policy may be one of those factors hindering taking full advantage of potential tax benefits. Table 6 highlights the implications on these conclusions of modelling the collaterals value as a GBM (Note 21):

Table 6. Optimal debt level and OAS for different L when the collateral's value is modelled as a GBM

L = 100%			
D	**TS**	**VL**	**OAS**
50.0	1.419	100.67	0.38%
60.0	2.021	100.97	1.04%
65.0	2.543	101.35	1.70%
70.0	**3.101**	**101.58**	**2.66%**
75.0	3.230	101.28	4.36%
80.0	2.540	99.61	8.59%
L = 30%			
D	**TS**	**VL**	**OAS**
35.00	**0.903**	**100.85**	**0.05%**
50.00	1.357	100.61	0.38%
60.00	1.313	100.26	1.04%
65.00	1.289	100.09	1.70%
70.00	1.197	99.68	2.66%
75.00	1.056	99.11	4.36%
80.00	0.814	97.88	8.59%

Modelling collateral as a GBM does not fundamentally change the optimal debt level and the consequences of the different macro tax policies discussed but erroneously diminished the perceived total risk of the debt expressed through the OAS and the prepayment fee. GBM may lead to severe mispricing risk undercovering on the side of financial institutions. Again, this is a potential threat to the FI's stability and profitability.

The repayment structure of loans has often been mentioned as being relevant to mitigate the credit risk for any given level of debt as for example in EBA/GL/2020/06. But transaction structure not only impacts credit risk but also, in combination with tax deductibility, risk taking incentives which might be left hidden in the loan underwriting process if not explicitly addressed. Table 7 shows the optimal debt level and corresponding OAS when the loan is interest only.

Table 7. Optimal debt level and OAS for a 5-year interest only loan. Other inputs as in Table 1

LTV = D	**TS**	**VL**	**OAS**
30.00	0.845	100.78	0.142%
35.00	1.052	100.82	0.309%
40.00	**1.248**	**101.01**	**0.413%**
45.00	1.400	100.88	0.841%
50.00	1.379	100.75	1.160%
55.00	1.313	100.34	1.627%

Eliminating interim amortizations has two clear effects when compared with the results of Table 2, (i) for all levels of debt the price of prepayment and credit risk is significantly higher and (ii) seeking maximum firm value through a larger interest tax shield, the equity investor has incentives to increase the level of debt by 5 percentage points.

It is not only the transaction structure, but also the economic structure of a given jurisdiction that does influence risk taking through the tax system's positive bias towards debt and through the cost (in time and money) of the recovery process upon default. Our model includes as base case a proportional bankruptcy cost element which weighs on the final recovery of the lender and the higher the percentage bankruptcy cost, the lower the recovery rate and therefore the higher the OAS needed to compensate for the additional cost of risk. Table 8 shows the impact on the optimal debt level and its OAS when bankruptcy costs are 20% of the collaterals value:

Table 8. Optimal debt level and OAS for BC = 20%

LTV = D	TS	VL	OAS
30.00	0.797	100.67	0.127%
35.00	**0.955**	**100.72**	**0.205%**
40.00	1.160	100.66	0.386%
45.00	1.347	100.63	0.579%
50.00	1.327	100.45	1.042%

The results show no change in the optimal level of debt when compared to Table 2 but a significantly higher risk, expressed through the OAS, the higher the LTV. But a strictly proportional percentage level of BC seems too much of a simplification since most lending institutions have a minimum fixed workout costs and legal procedures also posit a minimum cost regardless of the value of the recovered collateral. Table 9 shows the optimal level of debt and OAS when

$$BC = \text{Max}\,(FBC;\ V\,(i,t)\,(1 - VBC)) \qquad (9)\ (\text{Note } 22)$$

Where, FBC is a fixed minimum bankruptcy cost (in our numerical experiment FBC = 10) and VBC a variable percentage cost.

Table 9. Optimal debt level and OAS for BC = Max (FBC; V (1- VBC)) and FBC = 10

D	TS	VL	OAS
30.00	0.826	100.83	0.212%
35.00	0.941	100.99	0.299%
40.00	1.218	101.22	0.516%
45.00	**1.359**	**101.36**	**0.731%**
50.00	1.329	101.33	1.149%
55.00	1.303	101.30	1.582%

When compared to Table 2, the optimal level of debt has climbed from 35% to 45% and the equilibrium spread has moved from 0.315% to 0.731%. Not a surprise, a costlier and less efficient recovery process is not only increasing risk costs in the system but also incentivizing financial risk taking! (Note 23) When combined with a full deductibility of interests' policy the effects have a very strong impact on risk cost, severity and risk-taking incentives as shown in Table 10:

Table 10. Optimal debt level and OAS for BC = Max (FBC; V (1-BC)), FBC = 10 and L = 100%

LTV = D	TS	VL	OAS
50.00	1.777	101.78	1.149%
55.00	2.163	102.12	1.625%
60.00	2.680	102.57	2.261%
65.00	**3.314**	**102.91**	**3.397%**
70.00	3.240	102.30	5.366%
75.00	3.275	102.25	7.278%

Such simple macro policy errors shift the optimal debt level from 35% (Table 2) to 65% (Table 9) while risk costs through severity are increased by almost 40 bp for a modest leverage of 50%. "Emerging" market or underdeveloped market premiums seem clearly justified and signal clear directions for macroprudential policies and guidelines.

3. Theoretical Background and Methodology

3.1 Structural Models of Debt Value and Credit Risk Valuation

The approach of valuing financial instruments as contingent claims on the firm value can be traced back to Black and Scholes (1973) who consider equity as a European call on the assets of the firm with strike price equal to the firm's debt as one of the applications of their European call option model. Merton (1974) derives the value of the zero-coupon debt under the same assumptions and modelling framework. Many others have extended the framework to include coupons (Kim, Ramaswamy, Sundaresan, 1993), early default (Black and Cox, 1976), stochastic interest rates and exogenous recovery amounts (Longstaff and Schwartz, 1995), stochastic interest rates and early default (Brys and Varenne, 1995), possible strategic debt service (Anderson and Sundaresan, 1996). The compound option model of Geske (1979) allows for non- constant volatility. All these models share that assumption that the unlevered firm value follows a GBM, which, as already stated, is not in line with empirical evidence since firm value returns or, as in our case, real estate returns do not exhibit normally distributed returns. To overcome this difficulty, Zhou (2001), introduces jumps into the asset value stochastic process and finds a semi- closed form solution for the debt value and obtains credit spreads which are closer to empirical evidence.

3.2 Structural Models of Debt Value and Optimal Capital Structure

The problem of the existence of an optimal capital structure can be traced back at least to Salomon (1953) where he describes what is known as the traditional position: the optimal leverage is obtained on the point where the weighted average cost of capital (Wacc) is minimized. Implicitly, he states that for certain levels of debt, the Wacc starts growing again due to the existence of bankruptcy costs which is in contrast with the Modigliani Miller (1963) firm's value concept that only includes interest tax shields. (Note 24) Since then, such Trade Off Theory (Note 25) of the optimal capital structure has been investigated and models provided by Robicheck and Myers (1966), Kraus and Litzenberger (1973) in a state preference model and Scott (1976) by discounting flows subject to some default probabilities. Brennan and Schwartz (1978) where the first to apply a contingent claims framework to the optimal capital structure problem and where followed by the important papers of Leland (1994) and Leland and Toft (1996) where the optimal capital structure and the value of perpetual debt (Leland, 1994) is analyzed in a continuous time framework extended by Leland and Toft (1996) for constant, finite maturity debt. Empirical support for bankruptcy costs can be found in the classic paper of Warner (1977), Altman (1984), Andrade and Kaplan (1997) who estimate total bankruptcy costs at between 10% and 23%, Davydenko (2011), who estimate total bankruptcy costs at close to 22% and Reindl, Stoughton and Zechner (2017) who arrive at an estimate of around 20%. Cannaday and Yang (1996) and McDonald (1998) derive optimal capital structure models with trade off elements applied to real estate investments in a deterministic framework, while Gao and Wang (1990) propose an empirical model to test capital structure decisions in real estate investments finding some support for the Trade Off Theory. The Trade Off Theory remains one of the dominant theories of capital structure decisions with empirical support also for non- real estate firms in the classic papers of Bradley, Jarrell and Kim (1983), Rajan and Zingales (1995), Hovakimian, Opler and Titman (2001) and more recently in De Angelo et al (2011), Strebulaev and Whited (2012) and Dierker et al (2019)

Our analysis parts from Brennan and Schwartz (1978) in the sense that we propose a discrete time model, where we assume that the tax advantage of debt is lost in case of default and the collateral value follows a shifted lognormal distribution as in Haahtela (2006 and 2012) to better capture the skewness and kurtosis of the collateral value distribution. Additionally, our model obtains endogenously the optimal capital structure by the iteration of different scenarios of debt amount (ie. LTV), bankruptcy costs and interest tax shield. We introduce a default trigger that includes not only the asset value falling below the value of the debt but also the event of the net income of the property not being able to fully cover the debt service due (Note 26). Additionally, we are considering that the interest tax shields are capped up to a certain level of the income generated by the assets. Finally, we are assuming the debt is amortizing so debt service includes not only interests but also partial principal repayment.

As in De Luna et al (2025), we implement Haathela's shifted lognormal asset value model (Haahtela, 2006 and Camara and Chung, 2006) in a binomial tree that adds significant flexibility to account for realistic features of loan agreements (interim amortization payments inter alia). The stochastic process of Haahtela's displaced lognormal distribution is as follows:

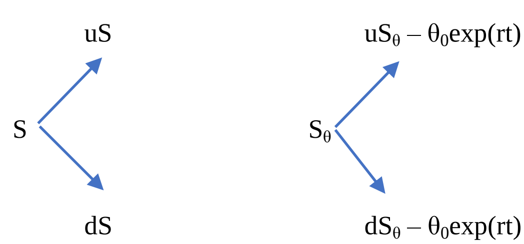
$$S_T = S_{\theta;0} e^{\left(\left(\mu - \frac{1}{2}\sigma_d^2\right)T + \sigma_d \sqrt{T} z\right)} + \theta_0 e^{\mu T} \; with \; z \sim N(0,1)$$

$$E(S_t) = (S_{\theta;0} + \theta_0) e^{\mu t}$$

$$S_{\theta,t} = S_{\theta,0} = e^{\left(\mu - \frac{\sigma_d^2}{2}\right)t + \sigma_d \sqrt{t} z_t}$$

$$std(S_t) = \left|S_{\theta;0}\right| e^{\mu t} \sqrt{e^{\sigma_d^2} - 1} \qquad (10)$$

The resulting distribution has heavier tails that the lognormal distribution better reflecting empirical stylized facts regarding tail events, while the volatility is no longer constant, (Note 27) again a convenient characteristic. This process can be easily approximated as in De Luna et al (2025) and Haahtela (2006) by the following binomial tree:



Where,

Θ is the constant shifting parameter

S_θ is the stochastic part of S_t

S(t) is the value of the collateral property

σ_d is the volatility of S_θ

With the parameters of the binomial tree by Wilmott et al (1995):

$$u = e^{((r-\delta)\Delta t)}\left(1 + \sqrt{e^{\sigma_d{}^2\Delta t}}\right) - 1$$

$$d = e^{((r-\delta)\Delta t)}\left(1 - \sqrt{e^{\sigma_d{}^2\Delta t}}\right) - 1$$

$$p = 0.5$$

In practical applications θ and σ_d may help calibrate the tree to market data or historical time series of returns and values.

4. Conclusions

Combining a modified and extended Modigliani Miller (1958 and 1963) firm value approach with (real) option pricing theory, we have developed, and implemented through numerical experiments with plausible parameters, a structural firm value model for the valuation of commercial real estate backed loans or bonds that simultaneously derives the optimal debt level. The purpose is not only to locate the optimal amount of debt as a percentage of the collateral value, but also to derive some macro prudential implications for lenders, investors and supervisors. The existence of an interior solution to the optimal capital structure problem is derived from the trade-off between interest tax shield (ITS or TS) increasing the levered firm value and bankruptcy or leverage costs (BC) reducing it. The optimal debt level is achieved where ITS – BC is maximal. The numerical implementation via a binomial tree can deal with any kind of principal repayment and interest payments while simultaneously obtaining the market value of the loan, the interest tax shield, the bankruptcy costs and the levered firm. Our approach, in the spirit of De Luna et al (2025) although with a different default condition and adding the valuation of credit risky interest tax shields and bankruptcy cost, includes (a) the shifted lognormal distribution of collateral value returns of Haahtela (2006) which more faithfully captures skewness and kurtosis of asset value returns, (b) a (legal) limitation on the tax deductibility of interests and (c) an analysis of several conceptualizations of bankruptcy costs to analyze its impact on the location of the optimal amount of debt. In our base case, and for a 5- year loan, the optimal amount of debt is significantly smaller than those implied by benchmark models (Leland (1994) and Leland and Toft (1996)). Some of the factors determining the low level of optimal debt may help reconsider the capital structure puzzle and challenge conventional wisdom that firms in general are underleveraged. We find that (i) as in De Luna et al (2025) and under suitable values of the shifting parameter, credit spreads resemble empirical evidence and are significantly higher than for the case of a lognormally distributed collateral (ii) incorrectly modelling the collateral value as a GBM produces the wrong impression of a reduced price of credit risk for typical leverage levels. This highlights the necessity to tailor the model to the specific return distribution characteristics of each property type and location if credit risk mispricing is to be avoided (iii) legally limiting the interest tax deductibility reduces the optimal amount of debt limiting therefore risk taking incentives (with no limitation, the optimal debt can double the one with a 30% limitation) and macroprudential risks (iv) developing a cost efficient and speedy work out and resolution legal framework strongly impacts the optimal debt level and the price of credit risk and should also be included in macroprudential frameworks (iv) the repayment structure of the transaction impacts not only the loan’s price of risk but also the optimal capital structure and the risk- taking incentives. Our conclusion is that each specific combination of collateral value risk, interest tax deductibility, bankruptcy cost function and transaction structure in each legal environment have their unique optimal debt level and price of risk. Consequently, simplified credit risk management or underwriting approaches based on simply monitoring LTV are mis specified and may lead to severe loan mispricing and hidden risk- taking by the lenders. A given LTV may be high risk for a specific property collateral in one location and low risk for a different property in another location. Considering and controlling these drivers may allow regulators to limit risk taking incentives by avoiding combination of drivers that lead to too high LTV situations relative to their optimal level. Credit ratings should reflect all these factors and not only debt level and initial debt service coverage. Our model may also help lenders and investors to gain full awareness of the risk-reward combinations of any given transaction. Correctly modelling collateral value risk, prudently defining interest tax deductibility and developing an advanced, efficient and speedy work out system should be of major concern for policy makers, lenders and investors alike. Our model does not address the optimal maturity of debt nor the impact of different tenors and roll over effects, which are left for future research.

Appendix A. R Markdown Script with Code Implementation

The following script implements a structural model for valuing real estate–backed loans and bonds. The framework combines a modified version of Modigliani and Miller (1958, 1963) with real options theory to capture the interaction between interest tax shields, bankruptcy costs and collateral value dynamics.

The model allows us to compute the levered firm value, the loan price, the interest tax shield, bankruptcy costs, and the optimal debt level.

Key features of the model include:

(i) The trade-off between tax benefits of debt and bankruptcy costs, where the optimal debt level maximizes their difference.

(ii) Collateral value distribution modeled using a shifted lognormal process.

(iii) Legal limitations on interest deductibility, which reduce both optimal debt level and risk-taking incentives.

(iv) Different specifications of bankruptcy costs and their impact on leverage.

(v) The structure of loan payments, which affects both risk exposure and optimal debt.

The main insight is that traditional conventional models tend to overestimate the level of optimal debt. Ignoring realistic collateral dynamics may additionally lead to underestimated risk. Therefore, risk assessment should not rely solely on loan-to-value (LTV) ratios but must also consider property type stochastic dynamic, loan structure, tax regulations, and bankruptcy costs. This model provides a richer framework for regulators, lenders, and investors to evaluate risk and determine sustainable debt levels.

1. Required Libraries

The code begins by installing and loading essential packages and libraries:

```
#install.packages(c("ggplot2", "viridis", "nleqslv", "reshape2"))

library(nleqslv)
library(reshape2)
```

2. Define the Range of Values for Parameters

Several variables are defined with specific ranges for different loan scenarios:

theta_values: Affects the shift of the lognormal distribution of the collateral property values. *sigma_values*: Represents the volatility of the property value. *term_values*: Represents the loan term (in years). *LTV_values*: Represents the Loan-to-Value ratio.

```
# Define parameter values for theta, sigma, loan term, LTV etc.

theta_values <- c(0, 10)      # Shift in collateral value distribution
sigma_values <- c(0.20)       # Volatility of the collateral (shifted lognormal distribution parameter)
term_values    <- c(5)           # Loan maturity in years
LTV_values      <- seq(0.25, 0.80, by = 0.05)    # Loan-to-value ratios to test optimal levels

Tx_L_mult_values <- c(1, 0.3)      # Multipliers for tax shield limit (full vs. partial deductibility of interests)
PPF_values          <- c(0.015, 0)    # Prepayment penalty (positive vs. no penalty)
BC_values            <- c(0.15, 0.20) # Bankruptcy costs (as fraction of collateral value)
FBC_values           <- c(0, 10)         # Fixed bankruptcy costs (absolute amount)
IPA_values           <- c(0, 0.02)      # Scheduled principal amortization rate per annum
q_values             <- c(0, 0.01)      # Suboptimal (“surprise”) default probabilities per period
r_values            <- c(0.0225, 0.0325) # Risk-free interest rates
Vu <- 100                                          # Value of the unlevered firm (base asset value)
# ---------------------------------------------------------------------------
# Goal: Compute the optimal loan spread given a set of parameters
#
# Description:
```

```r
# The function `calculate_spread` estimates the loan spread that equates
# the value of the loan with its nominal value, considering:
#      - Collateral dynamics ((shifted) binomial tree for asset value).
#      - Bankruptcy costs (fixed and proportional).
#      - Tax shield benefits (ITS tree).
#      - Prepayment penalties and default risk.
#      - Loan-to-Value (LTV), interest rates, and amortization.
#
# It works by:
# 1. Building asset value trees (with/without shift- theta-).
# 2. Simulating loan cash flows (interest, amortization, prepayment).
# 3. Constructing Mortgage Value Tree (MT3) and Interest Tax Shield Tree (MT4).
# 4. Using the root solver `nleqslv` to find the spread that equates the mortgage loan market value and its nominal or face value.
#
# Finally, nested loops run across all parameter combinations to build
# a comprehensive results table with spreads, MT3, and MT4 values.
# ----------------------------------------------------------------------------


calculate_spread <- function(theta, sigma, term, LTV, r, BC, q, Tx_L_mult, PPF, FBC, IPA) {

  # -----------------------------
  # Step 1: Define base parameters
  # -----------------------------

  periods_per_year <- 4
  n <- term * periods_per_year
  payment_steps <- term * periods_per_year
  delta_t <- term / n
  di <- 0.02
  r_minus_di <- r - di
  p <- 0.5
  exp_r_delta_t <- exp(r * delta_t)
  u <- exp(r_minus_di * delta_t) * (1 + sqrt(exp(sigma^2 * delta_t) - 1))
  d <- exp(r_minus_di * delta_t) * (1 - sqrt(exp(sigma^2 * delta_t) - 1))
  S_t <- 100
  S_theta <- S_t + theta
  Tx <- 0.25
  mu <- 0.04 * delta_t
  Amount <- LTV * S_t
  Tx_L <- Tx_L_mult * mu

  row <- n + 1
  col <- row

  # ---------------------------
  # Collateral Tree (S_theta)
  # ---------------------------

  matrix_tree <- matrix(NA, nrow = row, ncol = col)

  matrix_tree[1, 1] <- S_theta
  matrix_tree[2, 1] <- u * S_theta
  matrix_tree[2, 2] <- d * S_theta
```

```r
for (i in 3:row) {
   matrix_tree[i, 1] <- matrix_tree[i - 1, 1] * u
   for (j in 2:i) matrix_tree[i, j] <- matrix_tree[i - 1, j - 1] * d
}

# ---------------------------
# Adjusted Tree (S_t)
# ---------------------------

matrix_tree2 <- matrix(NA, nrow = row, ncol = col)

matrix_tree2[1,1] <- S_t
for (i in 2:row)
   for (j in 1:i)
      matrix_tree2[i,j] <- matrix_tree[i,j] - theta * exp(r*(i-1)*delta_t)

last_MT3 <- NULL
last_MT4 <- NULL

# The calculate_difference function is used within the spread calculation to find
# the difference between the calculated spread and the actual loan performance.

calculate_difference <- function(Spread) {

   Total_interest <- r + Spread
   row_tab <- 7
   col_tab <- payment_steps + 2
   table <- data.frame(
      `1` = c("Year","Tax shield limit (L)","Principal YE", "Interest", "IPA", "Period payment", "Prepay value (PPV)"),
      `2` = c(NA, NA, Amount, NA, NA, NA, NA),
      matrix(0, nrow = row_tab, ncol = col_tab-2)
   )

   # ---------------------------
   # Loan Cashflow Table
   # ---------------------------

   table[3,2] <- Amount
   for (j in 3:col_tab) {
      table[1,j] <- delta_t*(j-2)
      table[2,j] <- Tx_L*Vu
      table[5,j] <- Amount*IPA*delta_t
      table[3,j] <- table[3,j-1] - table[5,j]
      table[4,j] <- table[3,j-1]*Total_interest*delta_t
      table[6,j] <- table[4,j] + table[5,j]
      table[7,j] <- (table[3,j]+table[4,j]+table[5,j])*(1+PPF)
   }

   colnames(table) <- c("", "0", as.character(1:(col_tab-2)))
   table[row_tab, col_tab] <- NA
   table[5, col_tab] <- table[3,col_tab-1]
   table[6, col_tab] <- table[4, col_tab] + table[5,col_tab]
   table[3, col_tab] <- table[3, col_tab-1] - table[5,col_tab]
```

```
RR <- 0.8
q_adj <- q * delta_t

# The matrix matrix_tree3 (Mortgage Loan Value Tree) is updated using backward
# induction, where each node's value is recalculated based on previous nodes.

 # ---------------------------
 # Mortgage Tree
 # ---------------------------

matrix_tree3 <- matrix(NA, nrow = row, ncol = col)

for (j in 1:col) {
  if (table[6,col_tab] > matrix_tree2[row, j]) {
    calculated_value <- min(matrix_tree2[row, j] - FBC, matrix_tree2[row, j] *(1-BC))
  } else {
    calculated_value <- table[6,col_tab]
  }
  matrix_tree3[row, j] <- calculated_value
}

for (i in (row - 1):2) {
  for (j in 1:i) {
    if ((mu * matrix_tree2[i, j] < table[6, i+1]) & ((table[3, i+1] + table[6, i+1]) > matrix_tree2[i, j])) {
      calculated_value <- min(matrix_tree2[i, j] - FBC, matrix_tree2[i, j] * (1 - BC)) * (1 - q_adj)
    } else {
      calculated_value <- (
        min(
          table[7, i+1],
          (matrix_tree3[i + 1, j] * p + matrix_tree3[i + 1, j + 1] * (1 - p)) / exp_r_delta_t + table[6, i+1]
        ) * (1 - q_adj) ) + (q_adj * (table[3, i+1] + table[5, i+1]) * RR)
    }
    matrix_tree3[i, j] <- calculated_value
  }
}

matrix_tree3[1, 1] <- (matrix_tree3[2, 1] * p + matrix_tree3[2, 2]    * (1 - p)) / exp_r_delta_t

last_MT3 <<- matrix_tree3

# This section builds the Interest Tax Shield (ITS) tree.
# Each node in matrix_tree4 represents the value of the tax shield at that
# point in time, accounting for interest deductibility limits, loan payments,
# and the possibility of default.

# ---------------------------
# ITS Tree
# ---------------------------

matrix_tree4 <- matrix(NA, nrow = row, ncol = col)

for (j in 1:col) {
  if ((table[3, col_tab] + table[6, col_tab]) > matrix_tree2[row, j]) {
    calculated_value_1 <- 0
```

```
          } else {
            calculated_value_1 <- min(table[4, col_tab] * Tx, table[2, col_tab])
          }
          matrix_tree4[row, j] <- calculated_value_1
        }

        for (i in (row - 1):2) {
          for (j in 1:i) {
            if (((table[3, i+1] + table[6, i+1]) > matrix_tree2[i, j]) & ((mu * matrix_tree2[i, j]) < table[6, i+1])) {
              calculated_value_1 <- 0
            } else {
              calculated_value_1 <- ((p * matrix_tree4[i + 1, j] + (1 - p) * matrix_tree4[i + 1, j + 1]) / exp_r_delta_t +
                                      min(table[2, i+1], table[4, i+1]) * Tx) * (1 - q_adj)
            }
            matrix_tree4[i, j] <- calculated_value_1
          }
        }

        matrix_tree4[1, 1] <- (matrix_tree4[2, 1] * p + matrix_tree4[2, 2]    * (1 - p)) / exp_r_delta_t

        last_MT4 <<- matrix_tree4

        return(table[3,2] - matrix_tree3[1,1])
      } # End of the calculate_difference function

      result <- nleqslv(x = 0, fn = calculate_difference,
                        control = list(ftol=1e-12, xtol=1e-12, maxit=100000))

      # The nleqslv solver is used to find the equilibrium spread (loan market value = loan face value)
      # the calculated spread is the equilibrium OAS for each set of parameters.


      optimal_Spread <- result$x
      return(list(Spread=optimal_Spread, Mortgage_Value_Tree=last_MT3[1,1], ITS_Tree=last_MT4[1,1], mu=mu))
} # End of the calculate_spread function

# ---- Define parameter vectors ----

# Initialize the final results table (empty at the start)

results_table <- data.frame()

# Nested loops: iterate through all parameter combinations

for (term in term_values) {                          # Different loan maturities
  for (LTV in LTV_values) {                            # Different Loan-to-Value ratios
    for (theta in theta_values) {                   # Collateral value shifts
      for (sigma in sigma_values) {                # Collateral volatility
        for (Tx_L_mult in Tx_L_mult_values) {    # Tax shield multipliers
          for (PPF in PPF_values) {               # Prepayment penalty factors
            for (BC in BC_values) {              # Bankruptcy cost rates
              for (FBC in FBC_values) {         # Fixed bankruptcy costs
                for (IPA in IPA_values) {   # Scheduled amortization rates
                  for (q in q_values) {        # Default probabilities
                    for (r in r_values) {    # Risk-free interest rates
```

```
                        # Call the function that calculates the optimal spread
                        # and returns both the Mortgage Value Tree and the ITS Tree

                        res <- calculate_spread(theta, sigma, term, LTV, r, BC, q,
                                                Tx_L_mult, PPF, FBC, IPA)

                        # Append the results for this parameter combination

                        results_table <- rbind(results_table, data.frame(
                          Term = term,
                          LTV = LTV,
                          Theta = theta,
                          Sigma = sigma,
                          Tx_L = res$mu * Tx_L_mult,      # Effective tax shield adjustment
                          PPF = PPF,
                          BC = BC,
                          FBC = FBC,
                          IPA = IPA,
                          q = q,
                          r = r,
                          Spread = res$Spread,             # Optimal spread
                          Mortgage_Value_Tree = res$Mortgage_Value_Tree, # Mortgage Tree [1,1]
                          ITS_Tree = res$ITS_Tree          # ITS Tree [1,1]
                        ))
                      }
                    }
                  }
                }
              }
            }
          }
        }
      }
    }
}

# Print the final results table with all parameter combinations

# 3072 rows and 14 columns
# print(results_table)

# show first 6 rows of results_table

head(results_table)
##   Term LTV Theta Sigma Tx_L PPF   BC  FBC  IPA  q      r       Spread
##     5  0.25   0   0.2 0.01 0.015 0.15  0  0.00 0.00 0.0225 0.0001288415
##     5  0.25   0   0.2 0.01 0.015 0.15  0  0.00 0.00 0.0325 0.0001880183
##     5  0.25   0   0.2 0.01 0.015 0.15  0  0.00 0.01 0.0225 0.0020935640
##     5  0.25   0   0.2 0.01 0.015 0.15  0  0.00 0.01 0.0325 0.0021801047
##     5  0.25   0   0.2 0.01 0.015 0.15  0  0.02 0.00 0.0225 0.0001085812
##     5  0.25   0   0.2 0.01 0.015 0.15  0  0.02 0.00 0.0325 0.0001478945

##    Mortgage_Value_Tree   ITS_Tree
```

```
## 25 0.6669046
## 25 0.9389334
## 25 0.7064926
## 25 0.9711555
## 25 0.6352148
## 25 0.8945173
# ------------------------------------------------------------------------------
# Recalculation of Mortgage Value Tree and Interest Tax Shield Tree
# assuming Bankruptcy Costs (BC) = 0.
#
# Steps:

# 1. Define the function `recalculate_matrices_with_spread()`:
#       - Rebuilds the binomial trees (collateral and adjusted).
#       - Uses previously calculated optimal spreads.
#       - Returns Mortgage and Interest Tax Shield values with BC = 0.
# 2. Apply this function to all cases in `results_table`.
# 3. Build `results_table_BC0` with the new values.
# 4. Construct a `comparison_table`:
#       - Original Mortgage Tree and Interest Tax Shield Tree vs BC=0 versions.
#       - Differences (Diff_MG, Diff_ITS).
#       - Firm value adjusted (Vl).
# ------------------------------------------------------------------------------

recalculate_matrices_with_spread <- function(theta, sigma, term, LTV, r, Spread,
                                             q, PPF, IPA, FBC, Tx_L) {

  BC <- 0 # Bankruptcy costs are set to 0
  periods_per_year <- 4
  n <- term * periods_per_year
  payment_steps <- term * periods_per_year
  delta_t <- term / n
  di <- 0.02
  r_minus_di <- r - di
  p <- 0.5
  exp_r_delta_t <- exp(r * delta_t)
  u <- exp(r_minus_di * delta_t) * (1 + sqrt(exp(sigma^2 * delta_t) - 1))
  d <- exp(r_minus_di * delta_t) * (1 - sqrt(exp(sigma^2 * delta_t) - 1))
  S_t <- 100
  S_theta <- S_t + theta
  Tx <- 0.25
  mu <- 0.04 * delta_t
  Amount <- LTV * S_t
  Tx_L <- Tx_L
  Vu <- 100

  row <- n+1
  col <- row

  # ---------------------------
  # Collateral Tree (S_theta)
  # ---------------------------

  matrix_tree <- matrix(NA, nrow=row, ncol=col)
  matrix_tree[1,1] <- S_theta
```

```
matrix_tree[2,1] <- u*S_theta
matrix_tree[2,2] <- d*S_theta

for(i in 3:row) {
  matrix_tree[i,1] <- matrix_tree[i-1,1]*u
  for(j in 2:i)
    matrix_tree[i,j] <- matrix_tree[i-1,j-1]*d
}

# --------------------------
# Adjusted Tree (S_t)
# --------------------------

matrix_tree2 <- matrix(NA, nrow=row, ncol=col)
matrix_tree2[1,1] <- S_t
for(i in 2:row) {
  for(j in 1:i) {
    matrix_tree2[i,j] <- matrix_tree[i,j]-theta*exp(r*(i-1)*delta_t)
  }
}


q <- q*delta_t
RR <- 0.8

# --------------------------
# Cashflow Table
# --------------------------

row_tab <- 7
col_tab <- payment_steps + 2
table <- data.frame(
  `1` = c("Year","Tax shield limit (L)","Principal YE", "Interest", "IPA", "Period payment", "Prepay value (PPV)"),
  `2` = c(NA, NA, Amount, NA, NA, NA, NA),
  matrix(0, nrow=row_tab, ncol=col_tab-2)
)
table[3,2] <- Amount
for(j in 3:col_tab){
  table[1,j] <- delta_t*(j-2)
  table[2,j] <- Tx_L*Vu
  table[5,j] <- Amount*IPA*delta_t
  table[3,j] <- table[3,j-1]-table[5,j]
  table[4,j] <- table[3,j-1]*(r+Spread)*delta_t
  table[6,j] <- table[4,j]+table[5,j]
  table[7,j] <- (table[3,j]+table[4,j]+table[5,j])*(1+PPF)
}
colnames(table) <- c("", "0", as.character(1:(col_tab-2)))
table[row_tab,col_tab] <- NA
table[5,col_tab] <- table[3,col_tab-1]
table[6,col_tab] <- table[4,col_tab] + table[5,col_tab]
table[3,col_tab] <- table[3,col_tab-1]-table[5,col_tab]

# --------------------------
# Mortgage Tree
```

```
  # ---------------------------

  matrix_tree3 <- matrix(NA, nrow=row, ncol=col)

  for(j in 1:col){
    if(table[6,col_tab] > matrix_tree2[row,j]) {
      calculated_value <- min(matrix_tree2[row,j]-FBC, matrix_tree2[row,j]*(1-BC))
    } else {
      calculated_value <- table[6,col_tab]
    }
    matrix_tree3[row,j] <- calculated_value
  }
  for(i in (row-1):2){
    for(j in 1:i){
      if((mu*matrix_tree2[i,j]<table[6,i+1]) & ((table[3,i+1]+table[6,i+1])>matrix_tree2[i,j])) {
        calculated_value <- min(matrix_tree2[i,j]-FBC, matrix_tree2[i,j]*(1-BC))*(1-q)
      } else {
        calculated_value <- (min(table[7,i+1], (matrix_tree3[i+1,j]*p +
matrix_tree3[i+1,j+1]*(1-p))/exp_r_delta_t + table[6,i+1])*(1-q)) + (q*(table[3,i+1]+table[5,i+1])*RR)
      }
      matrix_tree3[i,j] <- calculated_value
    }
  }

  matrix_tree3[1,1] <- (matrix_tree3[2,1]*p + matrix_tree3[2,2]*(1-p))/exp_r_delta_t

  # ---------------------------
  # ITS Tree
  # ---------------------------

  matrix_tree4 <- matrix(NA, nrow=row, ncol=col)

  for(j in 1:col){
    if((table[3,col_tab]+table[6,col_tab])>matrix_tree2[row,j]) {
      calculated_value_1 <- 0
    } else {
      calculated_value_1 <- min(table[4,col_tab]*Tx, table[2,col_tab])
    }
    matrix_tree4[row,j] <- calculated_value_1
  }
  for(i in (row-1):2){
    for(j in 1:i){
      if(((table[3,i+1]+table[6,i+1])>matrix_tree2[i,j]) & ((mu*matrix_tree2[i,j])<table[6,i+1])) {
        calculated_value_1 <- 0
      } else {
        calculated_value_1 <- ((p*matrix_tree4[i+1,j]+(1-p)*matrix_tree4[i+1,j+1])/exp_r_delta_t +
min(table[2,i+1],table[4,i+1])*Tx)*(1-q)
      }
      matrix_tree4[i,j] <- calculated_value_1
    }
  }

  matrix_tree4[1,1] <- (matrix_tree4[2,1]*p + matrix_tree4[2,2]*(1-p))/exp_r_delta_t

  # Return matrix_tree3 and matrix_tree4 with BC = 0
```

```
# The difference in value between the Mortgage Value tree with BC >0 and the Mortgage Value tree with BC=# 0 in [1,1] is the present value of the bankruptcy costs

  return(list(
    Term = term,
    LTV = LTV,
    Theta = theta,
    Sigma = sigma,
    Spread = Spread,
    r = r,
    PPF = PPF,
    IPA = IPA,
    FBC = FBC,
    q = q,
    Tx_L = Tx_L,
    Mortgage_Tree_BC0 = matrix_tree3[1,1],
    ITS_Tree_BC0 = matrix_tree4[1,1]
  ))
}

  # -----------------------------------------
  # Build results_table_BC0 with under BC = 0
  # -----------------------------------------

results_table_BC0 <- data.frame()

for(i in 1:nrow(results_table)){
  theta_i <- results_table$Theta[i]
  sigma_i <- results_table$Sigma[i]
  term_i <- results_table$Term[i]
  LTV_i <- results_table$LTV[i]
  spread_i <- results_table$Spread[i]
  Tx_L_i <- results_table$Tx_L[i]
  PPF_i <- results_table$PPF[i]
  FBC_i <- results_table$FBC[i]
  IPA_i <- results_table$IPA[i]
  q_i <- results_table$q[i]
  r_i <- results_table$r[i]

  matrices_BC0 <- recalculate_matrices_with_spread(theta_i, sigma_i, term_i, LTV_i, r_i, spread_i, q_i, PPF_i, IPA_i, FBC_i, Tx_L_i)

  results_table_BC0 <- rbind(results_table_BC0, data.frame(
    Term = term_i,
    LTV = LTV_i,
    Theta = theta_i,
    Sigma = sigma_i,
    Tx_L = Tx_L_i,
    PPF = PPF_i,
    FBC=   FBC_i,
    IPA = IPA_i,
    q = q_i,
    r = r_i,
    Spread = spread_i,
    Mortgage_Tree_BC0 = matrices_BC0$Mortgage_Tree_BC0,
```

```
    ITS_Tree_BC0 = matrices_BC0$ITS_Tree_BC0
  ))
}

  # 3072 rows x 13 columns
  # print(results_table_BC0)

  # show first 6 rows of results_table_BC0

  head(results_table_BC0)
##    Term    LTV Theta Sigma Tx_L    PPF   FBC   IPA      q        r            Spread
##       5   0.25       0     0.2 0.01 0.015   0    0.00 0.00 0.0225 0.0001288415
##       5   0.25       0     0.2 0.01 0.015   0    0.00 0.00 0.0325 0.0001880183
##       5   0.25       0     0.2 0.01 0.015   0    0.00 0.01 0.0225 0.0020935640
##       5   0.25       0     0.2 0.01 0.015   0    0.00 0.01 0.0325 0.0021801047
##       5   0.25       0     0.2 0.01 0.015   0    0.02 0.00 0.0225 0.0001085812
##       5   0.25       0     0.2 0.01 0.015   0    0.02 0.00 0.0325 0.0001478945

##     Mortgage_Tree_BC0 ITS_Tree_BC0
##            25.00375          0.6669046
##            25.00375          0.9389334
##             25.00357       0.7064926
##             25.00366       0.9711555
##             25.00454       0.6352148
##             25.00111       0.8945173
  # --------------------------------------------------
  # Build comparison table: original vs BC=0
  # --------------------------------------------------

comparison_table <- data.frame()

for(i in 1:nrow(results_table)){
  theta_i      <- results_table$Theta[i]
  sigma_i      <- results_table$Sigma[i]
  term_i       <- results_table$Term[i]
  LTV_i        <- results_table$LTV[i]
  spread_i    <- results_table$Spread[i]
  Tx_L_i       <- results_table$Tx_L[i]
  PPF_i        <- results_table$PPF[i]
  FBC_i        <- results_table$FBC[i]
  IPA_i        <- results_table$IPA[i]
  q_i          <- results_table$q[i]
  r_i          <- results_table$r[i]

  # Original Mortgage Tree and ITS Tree (with BC != 0)

  Mortgage_Value_Tree <- results_table$Mortgage_Value_Tree[i]
  ITS_Tree <- results_table$ITS_Tree[i]

  # Recalculate with BC = 0

  MT_BC0 <- recalculate_matrices_with_spread(
    theta      = theta_i,
    sigma      = sigma_i,
```

```r
    term       = term_i,
    LTV        = LTV_i,
    r          = r_i,
    Spread     = spread_i,
    q          = q_i,
    PPF        = PPF_i,
    IPA        = IPA_i,
    FBC        = FBC_i,
    Tx_L       = Tx_L_i
  )

  Mortgage_Tree_BC0 <- MT_BC0$Mortgage_Tree_BC0
  ITS_Tree_BC0 <- MT_BC0$ITS_Tree_BC0

  # Differences

  diff_MG <- Mortgage_Tree_BC0 - Mortgage_Value_Tree
  diff_ITS <- ITS_Tree_BC0 - ITS_Tree

  # Store results

  comparison_table <- rbind(comparison_table, data.frame(
    Term         = term_i,
    LTV          = LTV_i,
    Theta        = theta_i,
    Sigma        = sigma_i,
    Spread       = spread_i,
    r            = r_i,
    Tx_L         = Tx_L_i,
    PPF          = PPF_i,
    FBC          = FBC_i,
    IPA          = IPA_i,
    q            = q_i,
    Mortgage_Value_Tree = Mortgage_Value_Tree,
    Mortgage_Tree_BC0    = Mortgage_Tree_BC0,
    diff_MG = diff_MG,
    ITS_Tree = ITS_Tree,
    ITS_Tree_BC0    = ITS_Tree_BC0,
    diff_ITS = diff_ITS
  ))
}

# Add levered    firm value, VL = Vu + PVTS -PVBC

comparison_table$Vl <- Vu + comparison_table$ITS_Tree - comparison_table$diff_MG

# Show final comparison

# 3072 rows x 18 columns
# print(comparison_table)

# show first 6 rows of comparison_table

head(comparison_table)
```

```
##     Term   LTV Theta Sigma          Spread      r     Tx_L    PPF    FBC    IPA      q
##        5   0.25        0      0.2 0.0001288415 0.0225 0.01 0.015    0    0.00 0.00
##        5   0.25        0      0.2 0.0001880183 0.0325 0.01 0.015    0    0.00 0.00
##        5   0.25        0      0.2 0.0020935640 0.0225 0.01 0.015    0    0.00 0.01
##        5   0.25        0      0.2 0.0021801047 0.0325 0.01 0.015    0    0.00 0.01
##        5   0.25        0      0.2 0.0001085812 0.0225 0.01 0.015    0    0.02 0.00
##        5   0.25        0      0.2 0.0001478945 0.0325 0.01 0.015    0    0.02 0.00


##     Mortgage_Value_Tree Mortgage_Tree_BC0 diff_MG    ITS_Tree ITS_Tree_BC0
##                 25                   25.00375        0.003746660 0.6669046 0.6669046
##                 25                   25.00375        0.003752434 0.9389334 0.9389334
##                 25                   25.00357        0.003572718 0.7064926 0.7064926
##                 25                   25.00366        0.003655757 0.9711555 0.9711555
##                 25                   25.00454        0.004535102 0.6352148 0.6352148
##                 25                   25.00111        0.001112853 0.8945173 0.8945173
##     diff_ITS      Vl
##          0       100.6632
##          0       100.9352
##          0       100.7029
##          0       100.9675
##          0       100.6307
##          0       100.8934
# ------------------------------------------------------------------------------
# Goal: Find the optimal LTV for each parameter combination.
#
# Steps:
# 1. Split the full comparison table into sub-tables, one for each LTV value.
#       (Each sub-table has the same number of rows, corresponding to the other params).
# 2. For each row position across sub-tables:
#       - Compare the levered firm value (Vl).
#       - Select the LTV that maximizes Vl.
# 3. Construct a new table `max_rows` with only the optimal LTV per row.
# ------------------------------------------------------------------------------

# -----------------------------
# Step 1: Split the big table
# -----------------------------

n <- nrow(results_table_BC0)    # total rows in results_table_BC0
chunk_size <- n/length(LTV_values) # number of rows per LTV

# Create indices to group rows by LTV

indices <- rep(1:length(LTV_values), each = chunk_size)[1:n]

# Split into list of tables by LTV

tables_list <- split(comparison_table, indices)

# Optional: create separate data frames table1, table2, ...

for(i in 1:length(tables_list)) {
```

```
  assign(paste0("table", i), tables_list[[i]])
}
# -----------------------------
# Step 2: Find optimal LTV
# -----------------------------

max_rows <- data.frame()

# Loop over rows (same row index across all sub-tables = same params except LTV)

for (row_i in 1:chunk_size) {

  # Extract Vl values across all LTV tables for the current row

  valores <- sapply(tables_list, function(tbl) as.numeric(tbl$Vl[row_i]))

  # Find the index of the maximum Vl

  idx_max <- which.max(valores)

  # Select the row from the table corresponding to the max LTV

  fila_max <- tables_list[[idx_max]][row_i, ]

  # Add the row to the final table

  max_rows <- rbind(max_rows, fila_max)
}

# -----------------------------
# Step 3: Final result
# -----------------------------

# `max_rows` contains, for each parameter combination,
# the row with the LTV that maximizes firm value (Vl).

# 256 rows x 18 columns
# print(max_rows)

# Printing specific rows of the results table
# Note: The following rows correspond to specific tables in the article
# Row 5     -> Table 6
# Row 133 -> Table 5
# Row 157 -> Table 10
# Row 193 -> Table 7
# Row 197 -> Table 2
# Row 205 -> Table 9
# Row 229 -> Table 3

print(max_rows[c(5, 133, 157, 193, 197, 205, 229), ])
##    Term   LTV   Theta Sigma      Spread          r       Tx_L   PPF   FBC IPA   q
##      5    0.70      0     0.2 0.026560159 0.0225 0.010 0.015      0 0.02 0
##      5    0.70     10     0.2 0.035727838 0.0225 0.010 0.015      0 0.02 0
##      5    0.65     10     0.2 0.033965411 0.0225 0.010 0.015     10 0.02 0
```

```
##       5     0.40       10      0.2 0.004125561 0.0225 0.003 0.015      0 0.00 0
##       5     0.35       10      0.2 0.001674257 0.0225 0.003 0.015      0 0.02 0
##       5     0.45       10      0.2 0.007306926 0.0225 0.003 0.015     10 0.02 0
##       5     0.40       10      0.2 0.007640245 0.0225 0.003 0.000      0 0.02 0

## Mortgage_Value_Tree Mortgage_Tree_BC0    diff_MG      ITS_Tree ITS_Tree_BC0
##              70                  71.51801         1.51800871 3.1007846 3.1007846
##              70                  71.67994         1.67993620 3.1355860 3.1355860
##              65                  65.40214         0.40214052 3.3139081 3.3139081
##              40                  40.23695         0.23694667 1.2481229 1.2481229
##              35                  35.17448         0.17448093 0.9408819 0.9408819
##              45                  45.00000         0.00000000 1.3586778 1.3586778
##              40                  40.07534         0.07534345 1.3262773 1.3262773
##       diff_ITS            Vl
##              0   101.5828
##              0   101.4556
##              0   102.9118
##              0   101.0112
##              0   100.7664
##              0   101.3587
##              0   101.2509
```

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Authors' contributions

Conceptualization: A.C.D.L.L., W.L.D.L.B, L.M.S; Methodology: A.C.D.L.L., W.L.D.L.B, L.M.S; Software: A.C.D.L.L.; Validation: A.C.D.L.L., W.L.D.L.B, L.M.S; Formal Analysis and Investigation: A.C.D.L.L., W.D.L.B, L.M.S; Resources and Data curation: A.C.D.L.L., W.D.L.B, L.M.S; Writing, Original Draft preparation: A.C.D.L.L., W.D.L.B, L.M.S; Review and Editing: A.C.D.L.L., W.D.L.B, L.M.S

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Data sharing statement

No additional data are available.

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Notes

Note 1. Maximizing Project value is equal to maximizing equity value if the face value of debt equals its market value.

Note 2. Proposition I (MM1958) was restated in MM (1963) as VL = D + E = Vu + PVTS where VL is the leveraged value of the firm, Vu the unlevered value of the firm and PVTS is the present value of interest tax shields, D the market value of debt and E, the market value of equity. According to this equation, the optimal leverage (D/VL) ratio is 1, in stark contrast with empirical evidence (see Graham (2002)).

Note 3. Under the restrictive assumptions of Modigliani Miller (1963) of risk-free constant debt, PVTS = TD.

Note 4. Under the cost of capital approach initiated by Modigliani and Miller (1958) VL = D + E = FCF/Ku, where FCF is the free cash flow of the firm and Ku is the cost of unlevered equity. If risky debt and interest tax shield are to be considered, then VL = D + E = FCF/WACC, where WACC is weighted average cost of capital.

Note 5. PVBC is the present value of all bankruptcy costs. We call PVTS-PVBC the net tax shield (NTS).

Note 6. Scott (1976) develops a first model of this kind based on discounted cash flows under risk neutrality.

Note 7. Earliest papers on such a conceptualization are Robicheck and Myers (1968) and Kraus and Litzenberger (1973). Brennan and Schwartz (1978) directly model the optimal capital structure based on option pricing theory and the Static Trade Off concepts.

Note 8. The value of the asset at inception may be estimated based upon a Discounted Cash Flow method which inherits the paradigms of Modigliani Miller (1958 and 1963).

Note 9. The debt service obligation at any time t includes fees, interest payments any contractual amortization due at that specific moment in time.

Note 10. DSO is included in the MP at maturity.

Note 11. As opposed to the approach of De Luna A.C. et al (2025), which is focused on residential mortgages, we do expect the prepayment value or PPV(t) to include some fees as is customary for income producing real estate transactions.

Note 12. Commercial real estate backed loans typically do not include a constant payment formula but an interest plus a percent on principal amortization scheme.

Note 13. The model does not include stochastic interest rates. The assumption is of a fixed rate loan. The impact of this assumption is likely not critical. See De Luna et al (2025) and references therein.

Note 14. 5 years is a standard maturity for this type of loans. The model does not address the optimal debt maturity should it exist.

Note 15. Following Willmott et al (1995) parameterization.

Note 16. Commercial real estate research services offer quarterly time series of property type prices for many jurisdictions. Increasing the property price observation frequency is not "refining" the binomial tree as is the case for the continuous trading assumption, but an entirely new lending transaction structure since value or price steps would not coincide with mandatory payment steps (quarterly) and default would only be possible in the payment steps. This set up reduces considerably the tree implied PD and the OAS and may increase the optimal debt level. Modelling monthly asset value steps may be acceptable for residential mortgages to private individuals but less so for commercial real estate loans.

Note 17. Such optimization can be performed for example with Excel's Solver routine or with R optimizers as in Appendix A.

Note 18. Such an approach was followed by De Luna (2006).

Note 19. MM (1963) deals with perpetual debt while we are analyzing debt with a maturity of five years.

Note 20. The optimal amount of debt is higher because in Table 2, the prepayment fee is not considered tax deductible while in Table 3, the compensation for prepayment risk included in the OAS is tax deductible increasing the ITS and the optimal amount of debt.

Note 21. "Surprise" or unpredictable jumps to default as evidenced in some empirical papers are captures in our model by the variable q as in De Luna et al (2025). An upward shift in the risk- free rate does not modify the optimal level but reduces the OAS for all levels of debt.

Note 22. In the presence of fixed (FBC) and variable (VBC) bankruptcy costs, Equation 9 substitutes V(i,t) x (1-BC) in equations 3 and 5

Note 23. In the presence of fixed minimum bankruptcy costs and a shifted lognormal distribution of collateral returns, the mortgage value can exhibit negative values in some scenarios due to LGD occasionally exceeding 100%. Such a situation is not uncommon in practice and models should be able to capture such effects.

Note 24. Such a firm value concept is correct under the Modigliani Miller assumption of risk-free perpetual debt. The purpose of Modigliani Miller (1958, 1963) was not to find an optimal capital structure but to investigate the effect of leverage on the equity return (Proposition II). The famous capital structure irrelevance statement (Proposition I 1958) is only applicable under the strict assumptions of MM (no taxes, no bankruptcy costs, e.g. perfect markets).

Note 25. Other theories of the capital structure include the signaling theory (Ross, 1977), agency cost theory (Jensen and Meckling, 1976), the pecking order theory (Myers, 1984) and the market timing theory (Baker and Wurgler, 2002)

Note 26. Haffki and Henning (2025) find clear evidence on the relevance of cash flow metrics on default risk.

Note 27. As opposed to log-normal binomial tree of Cox, Ross and Rubinstein (1978)